

ON A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS OF EVEN ORDER

BY
JOAQUIN B. DIAZ

A Dissertation

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY IN APPLIED MATHEMATICS
AT BROWN UNIVERSITY, JUNE 1945.

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ON A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS OF EVEN ORDER.*

By JOAQUIN B. DIAZ.

1. Introduction.¹ In this paper, we consider linear partial differential equations of the form

$$(1.1) \quad L^N u = 0,$$

where $u = u(x, y)$ and L is a certain linear partial differential operator of the second order. The most important special case is that of

$$(1.2) \quad Lu = \Delta u + \phi(x)u_x + \psi(y)u_y,$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$; then (1.1) becomes

$$L^N u = \Delta^N u + \text{lower order terms} = 0.$$

Our main purpose is to construct a sequence of particular solutions of (1.1) having the following properties: (a) In the neighborhood of each point at which a solution of the equation $L^N u = 0$ is analytic, this solution can be expanded in a uniformly and absolutely convergent series of linear combinations of these particular solutions; (b) Each particular solution is defined and analytic in the whole plane and can be obtained by quadratures from the coefficients of L . Furthermore, all of these particular solutions are linear combinations of certain functions independent of N .

Our method is based upon the solution of partial differential equations with constant coefficients by the method of hypercomplex variables. [See Scheffers (3)², and Ketchum (2); Ward (5) gives an exhaustive bibliography on analytic functions of linear algebras.] This method involves replacing the given partial differential equation of higher order by a system of partial differential equations of the first order. Sobrero (4) has treated in detail the equation $\Delta^2 u = 0$ from this standpoint. For the equation

$$(1.3) \quad \Delta^N u = 0$$

the procedure is as follows. Formally, we determine the hypercomplex

* Received September 13, 1945.

¹ Presented to the American Mathematical Society, September 17, 1945. The problem was suggested to the author by Dr. Lipman Bers.

² Numbers in parenthesis refer to papers in the bibliography.

unit j_N so that every function $u = f(x, y) = f(x + j_N y)$ shall be a solution of (1.3). By direct substitution, setting $f^{(2N)} = \frac{d^{2N}f}{d^{2N}(x + j_N y)}$, we obtain

$$\Delta^N f = \sum_{l=0}^N \binom{N}{l} \frac{\partial^{2N} f}{\partial x^{2(N-l)} \partial y^{2l}} = f^{(2N)} \sum_{l=0}^N \binom{N}{l} j_N^{2l}.$$

Thus, the necessary and sufficient condition that every function $f(x, y) = f(x + j_N y)$ be a solution of (1.3) is that

$$(1.4) \quad (1 + j_N^2)^N = 0.$$

From (1.4), it is seen that any power of j_N greater than the $2N$ -th may be expressed as a linear combination of $1, j_N, \dots, j_N^{2N-1}$. The hypercomplex valued function $f(x, y) = \sum_{i=0}^{2N-1} a_i(x, y) j_N^i$ is said to be analytic at (x_0, y_0) provided there exists a neighborhood of (x_0, y_0) such that if (x_1, y_1) is any point of this neighborhood, the difference quotient

$$\frac{f(x, y) - f(x_1, y_1)}{(x - x_1) + j_N(y - y_1)}$$

approaches a limit independent of the mode of approach of the point (x, y) to the point (x_1, y_1) . Letting (x, y) approach (x_1, y_1) in the directions parallel to the x and y axes, and equating the results, we obtain

$$\partial f / \partial x = (1/j_N) (\partial f / \partial y);$$

that is

$$\sum_{i=0}^{2N-1} a_{i,x} j_N^i = (1/j_N) \sum_{i=0}^{2N-1} a_{i,y} j_N^i,$$

where $a_{i,x} = \partial a_i / \partial x$; etc. Multiplying through by j_N in the last equation and employing (1.4) in the form

$$j_N^{2N} = - \sum_{i=0}^{N-1} \binom{N}{i} j_N^{2i},$$

we obtain

$$\begin{aligned} & \sum_{k=0}^{N-1} [a_{2k-1,x} - a_{2N-1,x} \binom{N}{k}] j_N^{2k} + \sum_{k=0}^{N-1} a_{2k,x} j_N^{2k+1} \\ &= \sum_{k=0}^{N-1} a_{2k,y} j_N^{2k} + \sum_{k=0}^{N-1} a_{2k+1,y} j_N^{2k+1}. \end{aligned}$$

To simplify the writing, we set $a_{-1} \equiv 0$. This device of introducing zero terms will be used throughout, without explicit mention. Equating coefficients of like powers of j_N , we see that

$$(1.5) \quad \begin{aligned} a_{2k,x} &= a_{2k+1,y}, \\ a_{2k,y} &= a_{2k-1,x} - a_{2N-1,x} \binom{N}{k}, \end{aligned}$$

($k = 0, \dots, N-1$). The system (1.5) bears the same relation to (1.3) as the Cauchy-Riemann equations bear to Laplace's equation. To complete the analogy, it remains to show that each of the $2N$ functions a_i satisfies (1.3), and that given any solution u of (1.3), there is a solution a_i , $i = 0, \dots, 2N-1$, of (1.5) such that $a_0 = u$.

Returning now to (1.2), we rewrite L in the form

$$(1.6) \quad Lu = (\tau/\sigma) [(\sigma u_x/\tau)_x + (\sigma u_y/\tau)_y],$$

where

$$\sigma = \exp \left(\int \phi dx \right), \quad \tau = \exp \left(- \int \psi dy \right).$$

Actually, we shall consider the more general case

$$(1.7) \quad Lu = (\tau_1/\sigma_2) [(\sigma_1 u_x/\tau_1)_x + (\sigma_2 u_y/\tau_2)_y].$$

If σ_i and τ_i do not change sign, the σ_i 's have the same sign, and the τ_i 's have the same sign, then (1.7) can be transformed into (1.6) by the change of variables

$$\xi = \int (\sigma_2/\sigma_1)^{1/2} dx, \quad \eta = \int (\tau_2/\tau_1)^{1/2} dy.$$

We prefer to treat the general case because no additional difficulties arise and the formal part of our considerations remains valid even when the σ 's and τ 's change sign. In particular, this is true when L is of hyperbolic or of mixed type.

The system of first order equations connected with (1.1) is

$$(1.8) \quad \begin{aligned} \sigma_1(x) a_{2k,x} &= \tau_1(y) a_{2k+1,y} \\ \sigma_2(x) a_{2k,y} &= \tau_2(y) [a_{2k-1,x} - a_{2N-1,x} \binom{N}{k}], \end{aligned}$$

($k = 0, \dots, N-1$), which arises in a natural way from (1.5).

Bers and Gelbart (1) have studied the system of equations³

³ A system of equations of the form (1.9), with $\sigma_1 = \sigma_2 = \tau_1 = 1$, and which appears in the theory of compressible fluids, has been discussed independently by Professor S. Bergman, who introduced an operator analogous to the "formal powers" employed by Bers and Gelbart [1]. See S. Bergman, "The hodograph Method in the Theory of Compressible Fluids" (Supplement to R. V. Mises and K. O. Friedrichs,

$$(1.9) \quad \begin{aligned} \sigma_1(x)u_x &= \tau_1(y)v_y, \\ \sigma_2(x)u_y &= -\tau_2(y)v_x, \end{aligned}$$

which is obtained from the Cauchy-Riemann equations just as (1.8) is obtained from (1.5). By eliminating v from (1.9), we obtain the equation $Lu = 0$.

To simplify the writing, we use the following real functions

$$(1.10) \quad \begin{aligned} X(x) &= \int_0^x (1/\sigma_1(\xi)) d\xi, & Y(y) &= \int_0^y (1/\tau_1(\eta)) d\eta, \\ X^*(x) &= \int_0^x \sigma_2(\xi) d\xi, & Y^*(y) &= \int_0^y \tau_2(\eta) d\eta, \end{aligned}$$

and rewrite (1.8) in the form

$$(1.11) \quad \begin{aligned} a_{2k,X} &= a_{2k+1,Y}, \\ a_{2k,Y^*} &= a_{2k-1,X^*} - a_{2N-1,X^*} \binom{N}{k}, \end{aligned}$$

($k = 0, \dots, N-1$), where $a_{2k,X} = \partial a_{2k}/\partial X$, etc. We shall refer to (1.8), or (1.11), as the system $E(\Sigma, N)$, where Σ is the matrix of the coefficients of (1.8)

$$(1.12) \quad \Sigma = \begin{vmatrix} \sigma_1 & \tau_1 \\ \sigma_2 & \tau_2 \end{vmatrix}.$$

We remark that if $u = u(x, y)$ has continuous second derivatives, then $\partial^2 u/\partial X \partial Y = \partial^2 u/\partial Y \partial X$, but that in general $\partial^2 u/\partial X \partial X^* \neq \partial^2 u/\partial X^* \partial X$, etc. With the aid of (1.10), the operator L of (1.7) may be rewritten in the form

$$Lu = u_{XX^*} + u_{Y^*Y}.$$

Together with (1.8), we shall be led to consider the system

$$(1.13) \quad \begin{aligned} (1/\sigma_2(x))a_{2k,x} &= \tau_1(y)a_{2k+1,y}, \\ (1/\sigma_1(x))a_{2k,y} &= \tau_2(y)[a_{2k-1,x} - a_{2N-1,x} \binom{N}{k}], \end{aligned}$$

($k = 0, \dots, N-1$), which may be rewritten in the form

$$(1.14) \quad \begin{aligned} a_{2k,X^*} &= a_{2k+1,Y}, \\ a_{2k,Y^*} &= a_{2k-1,X} - a_{2N-1,X} \binom{N}{k}, \end{aligned}$$

Fluid Dynamics), mimeographed lecture notes, Brown University, 1942, especially pages 23-24; and "A Formula for the Stream Function of Certain Flows," *Proceedings of the National Academy of Sciences*, vol. 29 (1943), pp. 276-281, by the same author.

($k=0, \dots, N-1$). We shall refer to (1.13), or (1.14), as the system $E(\Sigma', N)$, where Σ' is the matrix of the coefficients of (1.13)

$$(1.15) \quad \Sigma' = \begin{vmatrix} (1/\sigma_2) & \tau_1 \\ (1/\sigma_1) & \tau_2 \end{vmatrix}.$$

It will be assumed throughout that the functions $\sigma_1(x)$, $\sigma_2(x)$, $\tau_1(y)$, and $\tau_2(y)$ are *positive analytic functions defined for all values of their respective real variables*, but it is clear that some of the results hold under weaker hypotheses. All the hypercomplex valued functions we shall deal with are defined on the (x, y) plane. Accordingly, we shall write $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ for a typical function, where $z = (x, y) = x + iy$ is a variable point of the (x, y) -plane and $a_i = a_i(x, y)$, and speak of the domain in the plane on which f is defined. By $|z|$ is understood the distance of the point z from the origin and by a domain is understood an open connected set.

In order to construct a function theory for (1.11), it is first necessary to study the algebra generated by the unit j_N of (1.4). This is done in Section 2. The real algebra of order $2N$ generated by j_N is shown to be isomorphic to a certain complex algebra of order N ; and a criterion for a number of the algebra to possess an inverse is given. In Section 3 (Σ, N)-monogenic functions are introduced. These are functions $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ such that the a_i satisfy the system $E(\Sigma, N)$, and they take the place of analytic functions. For these functions, the processes of (Σ, N)-differentiation and (Σ, N)-integration are defined. These two processes transform (Σ, N)-monogenic functions into (Σ', N)-monogenic functions [see (1.14) and (1.15)]. The process of "contraction," which transforms (Σ, N)-monogenic functions into ($\Sigma, N-M$)-monogenic functions, is treated in Section 4. The relations between (Σ, N)-, (Σ', N)-, (Σ, N)-, and (Σ', N)-monogenic functions are discussed in Section 6, where the matrices of coefficients Σ , and Σ' , are defined. In Section 5 it is proved that if $f(z)$ is (Σ, N)-monogenic, then the real functions a_{2k} satisfy the equation $L^N u = 0$. The process of (Σ, N)-integration enables us to construct special (Σ, N)-monogenic functions, called "formal powers," which bear the same relation to $E(\Sigma, N)$ as the powers of $x + iy$ bear to the Cauchy-Riemann equations. By means of these "formal powers," it is possible (Section 8) to characterize completely all (Σ, N)-monogenic functions for which one of the functions a_i vanishes identically ("null" functions). It is then proved (Section 9) that every analytic function satisfying $L^N u = 0$ is an "even component" of a (Σ, N)-

monogenic function determined to within a "null function." The main result concerning (Σ, N) -monogenic functions is the expansion theorem proved in Section 10, which states that a (Σ, N) -monogenic function may be expanded uniquely in a formal power series in the neighborhood of any point at which the function is (Σ, N) -monogenic. Using this expansion theorem, we construct the sequence of particular solutions of $L^N u = 0$ mentioned at the outset.

2. The algebras A_{2N} .

A. For $N = 1, 2, 3, \dots$ let A_{2N} denote the algebra of order $2N$ over the real field, whose elements are the polynomials

$$(2.1) \quad \sum_{i=0}^{2N-1} a_i j_N^i,$$

where the a_i are real numbers and the unit j_N satisfies the equation

$$(2.2) \quad (1 + j_N^2)^N = 0.$$

Equality is defined by postulating that

$$\sum_{i=0}^{2N-1} a_i j_N^i = \sum_{i=0}^{2N-1} b_i j_N^i,$$

if, and only if, $a_i = b_i$ for $n = 0, \dots, 2N - 1$.

Addition is defined by the identity

$$\sum_{i=0}^{2N-1} a_i j_N^i + \sum_{i=0}^{2N-1} b_i j_N^i = \sum_{i=0}^{2N-1} (a_i + b_i) j_N^i.$$

Multiplication is defined by the identity

$$\left(\sum_{i=0}^{2N-1} a_i j_N^i \right) \left(\sum_{i=0}^{2N-1} b_i j_N^i \right) = \sum_{i=0}^{4N-2} \left(\sum_{l+k=i} a_l b_k \right) j_N^i = \sum_{i=0}^{2N-1} c_i j_N^i,$$

where the c_i 's are obtained by employing the relation [see (2.2)]

$$(2.3) \quad j_N^{2N} = - \sum_{k=0}^{N-1} \binom{N}{k} j_N^{2k}.$$

For each N , the system possesses the identity element 1. If a number of A_{2N} has an inverse, then the inverse is unique.

Let x be an indeterminate over the real field. Then, for any $N = 1, 2, 3, \dots$

$$(2.4) \quad \sum_{k=0}^{N-1} a_{2k} x^{2k} + \sum_{k=0}^{N-1} a_{2k+1} x^{2k+1} \equiv \sum_{l=0}^{N-1} (\alpha_l + x \beta_l) (1 + x^2)^l,$$

where the real numbers α_l and β_l are uniquely determined by

$$(2.5) \quad \begin{aligned} \alpha_l &= \sum_{k=l}^{N-1} (-1)^{k-l} \binom{k}{l} a_{2k}, \\ \beta_l &= \sum_{k=l}^{N-1} (-1)^{k-l} \binom{k}{l} a_{2k+1}, \end{aligned}$$

($l = 0, 1, \dots, N-1$). Thus, we have

THEOREM 2.1. Every number $\sum_{i=0}^{2N-1} a_i j_N^i$ of A_{2N} may be expressed in a unique way in the form $\sum_{l=0}^{N-1} (\alpha_l + j_N \beta_l) (1 + j_N^2)^l$.

B. In what follows, we shall make use of the fact that j_N^2 has an inverse. In order to obtain this inverse, we need the binomial identity

$$(2.6) \quad \sum_{k=q-1}^{N-1} \binom{k}{q-1} = \binom{N}{q},$$

where $N = 1, 2, \dots$ and $1 \leq q \leq N-1$. Using (2.6), we verify that

$$(2.7) \quad 1/j_N^2 = - \sum_{k=0}^{N-1} (1 + j_N^2)^k.$$

C. Let the sequence of positive real numbers p_i be defined by the equations

$$(2.8) \quad \sum_{i+l=k} p_i p_l = 1, \quad p_0 = 1,$$

$k = 0, 1, 2, \dots$. It follows that $p_i = (-1)^i \binom{-\frac{1}{2}}{i}$, since $(\sum_{i=0}^{\infty} p_i x^i)^2 = 1/1-x$ for $|x| < 1$, but the exact values of the p_i will not be needed. We define

$$(2.9) \quad i_N = j_N \sum_{i=0}^{N-1} p_i (1 + j_N^2)^i$$

for $N = 1, 2, \dots$. Employing (2.7) and (2.9), we have

THEOREM 2.2. The numbers i_N and $-i_N$ of the algebra A_{2N} satisfy the equation $x^2 = -1$.

THEOREM 2.3. Every number $\sum_{k=0}^{N-1} (\alpha_k + j_N \beta_k) (1 + j_N^2)^k$ of A_{2N} may be expressed in a unique way in the form $\sum_{k=0}^{N-1} (\gamma_k + i_N \delta_k) (1 + j_N^2)^k$, where i_N is given by (2.9); the real numbers γ_k and δ_k are given by

$$(2.10) \quad \begin{aligned} \gamma_k &= \alpha_k, \\ \sum_{i+l=k} \delta_i p_l &= \beta_k, \end{aligned}$$

($k = 0, 1, \dots, N-1$), and the p_l are given by (2.8).

This may be seen at once by direct computation. The last result may be stated in another way. For $N = 1, 2, 3, \dots$ let B_N denote the algebra of order N over the complex field whose elements are the polynomials

$$\sum_{i=0}^{N-1} b_i \omega_N^i,$$

where the b_i are complex numbers and the unit ω_N satisfies the equation

$$(\omega_N)^N = 0.$$

Equality is defined by postulating that

$$\sum_{i=0}^{N-1} b_i \omega_N^i = \sum_{i=0}^{N-1} c_i \omega_N^i,$$

if, and only if, $b_i = c_i$ for $i = 0, 1, \dots, N-1$. Addition and multiplication are defined by the identities

$$(2.11) \quad \begin{aligned} \sum_{i=0}^{N-1} b_i \omega_N^i + \sum_{i=0}^{N-1} c_i \omega_N^i &= \sum_{i=0}^{N-1} (b_i + c_i) \omega_N^i, \\ \left(\sum_{i=0}^{N-1} b_i \omega_N^i \right) \left(\sum_{i=0}^{N-1} c_i \omega_N^i \right) &= \sum_{i=0}^{N-1} \left(\sum_{l+k=i} b_l c_k \right) \omega_N^i. \end{aligned}$$

Theorem 2.3 states that for $N = 1, 2, 3, \dots$ the algebra A_{2N} of order $2N$ over the real field is isomorphic to the algebra B_N of order N over the complex field. In this isomorphism, the number $\sum_{k=0}^{N-1} (\alpha_k + j_N \beta_k) (1 + j_N^2)^k$ of A_{2N} corresponds to the number $\sum_{k=0}^{N-1} (\gamma_k + i \delta_k) \omega_N^k$ of B_N , where the γ_k and δ_k are given by (2.10) and $i = j_1 = i_1$ is the ordinary complex unit.

D. Using the same notation as above, let

$$\sum_{k=0}^{N-1} a_{2k} j_N^{2k} + \sum_{k=0}^{N-1} a_{2k+1} j_N^{2k+1} = \sum_{k=0}^{N-1} (\alpha_k + j_N \beta_k) (1 + j_N^2)^k,$$

be a number of A_{2N} . Recalling (2.5) and (2.10), and keeping in mind that $p_0 = 1$ from (2.8), we have

$$(2.12) \quad \begin{aligned} \gamma_0 = \alpha_0 &= \sum_{k=0}^{N-1} (-1)^k a_{2k}, \\ \delta_0 = \beta_0 &= \sum_{k=0}^{N-1} (-1)^k a_{2k+1}. \end{aligned}$$

THEOREM 2.4. *The number $\sum_{i=0}^{2N-1} a_i j_N^i = \sum_{k=0}^{N-1} (\gamma_k + i_N \delta_k) (1 + j_N^2)^k$ of A_{2N} has an inverse if, and only if, the ordinary complex number $\gamma_0 + i\delta_0$ does not vanish.*

Proof. In view of the isomorphism between A_{2N} and B_N , it suffices to show that the asserted condition is necessary and sufficient for the number $\sum_{i=0}^{N-1} b_i \omega_N^i = \sum_{k=0}^{N-1} (\gamma_k + i\delta_k) \omega_N^k$ of B_N to have an inverse.

Suppose that $b_0 = \gamma_0 + i\delta_0 = 1/c_0$ and define the complex numbers c_i by

$$(2.13) \quad \begin{aligned} b_0 c_0 &= 1, \\ \sum_{l+k=i} b_l c_k &= 0, \end{aligned}$$

($i = 1, 2, \dots, N-1$). Then, by (2.11), $\sum_{i=0}^{N-1} c_i \omega_N^i$ is the desired inverse. Conversely, if $\sum_{i=0}^{N-1} b_i \omega_N^i$ has the inverse $\sum_{k=0}^{N-1} c_k \omega_N^k$ then, from (2.11), equations (2.13) hold and c_0 is the inverse of b_0 .

Taking (2.12) into account, it follows that the number $\sum_{i=0}^{2N-1} a_i j_N^i$ of A_{2N} has an inverse if, and only if,

$$(2.14) \quad \left[\sum_{k=0}^{N-1} (-1)^k a_{2k} \right]^2 + \left[\sum_{k=0}^{N-1} (-1)^k a_{2k+1} \right]^2 > 0.$$

THEOREM 2.5. *If a number $\sum_{i=0}^{2N-1} a_i j_N^i$ of A_{2N} which is the sum of even (odd) powers of j_N has an inverse, then its inverse is also the sum of even (odd) powers of j_N .*

Proof. Let

$$(2.15) \quad \begin{aligned} \sum_{k=0}^{N-1} a_{2k} j_N^{2k} + \sum_{k=0}^{N-1} a_{2k+1} j_N^{2k+1} &= \sum_{k=0}^{N-1} (\alpha_k + j_N \beta_k) (1 + j_N^2)^k \\ &= \sum_{k=0}^{N-1} (\gamma_k + i_N \delta_k) (1 + j_N^2)^k, \end{aligned}$$

in A_{2N} and let

$$(2.16) \quad \sum_{i=0}^{N-1} b_i \omega_N^i = \sum_{k=0}^{N-1} (\gamma_k + i\delta_k) \omega_N^k,$$

be the image of $\sum_{i=0}^{2N-1} a_i j_N^i$ in B_N .

We note that all the $a_{2k} = 0$ in (2.15) if, and only if, all the b_i are purely imaginary in (2.16). Analogously, all the $a_{2k+1} = 0$ in (2.15) if, and only if, all the b_i are real in (2.16).

For, if all $a_{2k} = 0$ in (2.15), then all $\alpha_k = 0$ from (2.5), and all $\gamma_k = 0$ from (2.10). Hence, all the b_i of (2.16) are purely imaginary. Conversely, if all the b_i of (2.16) are purely imaginary, then all $\gamma_k = 0$ in (2.16), all $\alpha_k = 0$ from (2.10), and all $a_{2k} = 0$ from (2.5). Similarly for the a_{2k+1} .

Let the inverse of $\sum_{i=0}^{N-1} b_i \omega_N^i$ be denoted by $\sum_{i=0}^{N-1} c_i \omega_N^i$. To complete the proof, it is merely necessary to notice that in view of (2.11) and (2.13), if all the b_i are real (purely imaginary), then all the c_i are also real (purely imaginary), and to apply the above reasoning to $\sum_{i=0}^{N-1} c_i \omega_N^i$.

E. THEOREM 2.6. *If $f = \sum_{i=0}^{2N-1} a_i j_N^i$ and $g = j_N f = \sum_{i=0}^{2N-1} b_i j_N^i$, then*

$$(2.17) \quad \begin{aligned} b_{2k} &= a_{2k-1} - \binom{N}{k} a_{2N-1}, \\ b_{2k+1} &= a_{2k}, \end{aligned}$$

$(k = 0, \dots, N-1)$, and

$$(2.18) \quad \begin{aligned} a_{2k} &= b_{2k+1}, \\ a_{2k+1} &= b_{2k+2} - \binom{N}{k+1} b_0, \end{aligned}$$

$(k = 0, \dots, N-1)$.

(2.17) follows by direct computation, and (2.18) is obtained from it by observing that for $k = 0$ the first equation of (2.17) gives

$$b_0 = -a_{2N-1}.$$

We remark that if $g = j_N f$, then $f = g/j_N$, since j_N has an inverse, by (2.14).

Let $f = \sum_{i=0}^{2N-1} a_i j_N^i$ be a number of A_{2N} . We shall use the following notation:

$$(2.19) \quad \begin{aligned} \operatorname{Re}[f] &= a_0, \\ \operatorname{Im}[f] &= a_1. \end{aligned}$$

THEOREM 2.7. *Let $f = \sum_{i=0}^{2N-1} a_i j_N^i$ be a number of A_{2N} , and define*

$$(2.20) \quad K_{N,k} = - \sum_{l=k+1}^N \binom{N}{l} j_N^{2(l-k)},$$

$$F_{N,k} = K_{N,k} f,$$

($k = 0, \dots, N-1$). Then

$$(2.21) \quad \operatorname{Re}[F_{N,k}] = a_{2k}, \quad \operatorname{Im}[F_{N,k}] = a_{2k+1},$$

($k = 0, \dots, N-1$).

Proof. From the definition (2.20), we have at once

$$(2.22) \quad K_{N,k} = j_N^2 K_{N,k+1} - j_N^2 \binom{N}{k+1},$$

($k = 0, 1, \dots, N-2$). The proof of the theorem now follows by an induction on k . First, $K_{N,k}$, for $k = N-1$, has the desired property (2.21), by (2.20) and (2.18). The induction is completed with the aid of these same two equations plus (2.22).

THEOREM 2.8. $K_{N,k}$ has an inverse in A_{2N} which is the sum of even powers of j_N .

Proof. The existence of the inverse follows immediately from (2.14), the definition of $K_{N,k}$, and the identity

$$- \sum_{l=k+1}^N \binom{N}{l} (-1)^{l-k} = \binom{N-1}{k},$$

($k = 0, \dots, N-1$). That $1/K_{N,k}$ is a sum of even powers of j_N is seen from theorem 2.5.

F. In later sections, we shall employ the concept of the "absolute value," or norm, of a number of A_{2N} . Let $\alpha = \sum_{i=0}^{2N-1} a_i j_N^i$ be a number of A_{2N} . The absolute value of α , denoted by $\|\alpha\|$, is a non-negative number such that

$$(2.23) \quad \|\alpha\|^2 = \sum_{i=0}^{2N-1} a_i^2.$$

From this definition, we have the "triangle inequality"

$$(2.24) \quad \left| \|\alpha\| - \|\beta\| \right| \leq \|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|.$$

If c is a real number, then

$$(2.25) \quad \|c\alpha\| = |c| \|\alpha\|.$$

There is a useful inequality which gives an upper bound for the absolute

value of the product of two numbers of A_{2N} . There exists a positive constant Γ_N such that [see (2), page 643]

$$(2.26) \quad \|\alpha\beta\| \leq \Gamma_N \|\alpha\| \|\beta\|,$$

for any α and β of A_{2N} . Therefore

$$(2.27) \quad \|\alpha^n\| \leq \Gamma_N^{n-1} \|\alpha\|^n.$$

G. Any hypercomplex number in the algebra A_{2N} may be written as a polynomial in j_N of degree at most $2N-1$. It is desirable to have a formula giving the powers of j_N greater than the $2N-1$ -st as sums of powers of j_N ranging from 0 to $2N-1$. We have

$$(2.28) \quad \begin{aligned} j_N^{2\nu} &= (-1)^{N+\nu+1} \sum_{k=0}^{N-1} \binom{\nu}{k} \binom{\nu-1-k}{N-1-k} j_N^{2k}, \\ j_N^{2\nu+1} &= (-1)^{N+\nu+1} \sum_{k=0}^{N-1} \binom{\nu}{k} \binom{\nu-1-k}{N-1-k} j_N^{2k+1}. \end{aligned} \quad (\nu = N, N+1, \dots)$$

Only the first formula of (2.28) need be proved, since the second is a consequence of the first.

The proof is by induction, using the binomial identity

$$(2.29) \quad \binom{\nu}{k-1} \binom{\nu-k}{N-k} + \binom{\nu+1}{k} \binom{\nu-k}{N-k-1} = \binom{\nu}{N-1} \binom{N}{k}$$

H. Expanding $(x + j_N y)^p$ by means of the binomial formula and replacing the powers of j_N above the $2N-1$ -st by powers of j_N from the 0-th to the $2N-1$ -st (using 2.28), we obtain

$$(2.30) \quad \begin{aligned} (x + j_N y)^p &= \sum_{k=0}^{N-1} \left\{ \binom{p}{2k} y^{2k} x^{p-2k} + \sum_{\nu=N}^{[p/2]} [(-1)^{N+\nu+1} \binom{p}{2\nu} \binom{\nu}{k} \binom{\nu-k-1}{N-k-1} y^{2\nu} x^{p-2\nu}] \right\} \\ &+ \sum_{k=0}^{N-1} \left\{ \binom{p}{2k+1} y^{2k+1} x^{p-2k-1} + \sum_{\nu=N}^{[(p-1)/2]} [(-1)^{N+\nu+1} \binom{p}{2\nu+1} \binom{\nu}{k} \binom{\nu-k-1}{N-k-1} \right. \\ &\quad \left. y^{2\nu+1} x^{p-2\nu-1}] \right\} j_N^{2k+1}. \end{aligned}$$

For later reference, we shall write down (2.30) for even and odd p

$$(2.31) \quad \begin{aligned} (x + j_N y)^{2q} &= \sum_{k=0}^{N-1} \left\{ \binom{2q}{2k} y^{2k} x^{2q-2k} + \sum_{\nu=N}^q [(-1)^{N+\nu+1} \binom{2q}{2\nu} \binom{\nu}{k} \binom{\nu-k-1}{N-k-1} \right. \\ &\quad \left. y^{2\nu} x^{2q-2\nu}] \right\} j_N^{2k} + \sum_{k=0}^{N-1} \left\{ \binom{2q}{2k+1} y^{2k+1} x^{2q-2k-1} + \sum_{\nu=N}^{q-1} [(-1)^{N+\nu+1} \right. \\ &\quad \left. \binom{2q}{2\nu+1} \binom{\nu}{k} \binom{\nu-k-1}{N-k-1} y^{2\nu+1} x^{2q-2\nu-1}] \right\} j_N^{2k+1}, \end{aligned}$$

$$\begin{aligned}
 (2.32) \quad (x + j_N y)^{2q+1} &= \sum_{k=0}^{N-1} \left\{ \binom{2q+1}{2k} y^{2k} x^{2q+1-2k} + \sum_{\nu=N}^q [(-1)^{N+\nu+1} \binom{2q+1}{2\nu} \right. \\
 &\quad \left. \binom{\nu}{k} \binom{\nu-k-1}{N-k-1} y^{2\nu} x^{2q+1-2\nu} \right\} j_N^{2k} + \sum_{k=0}^{N-1} \left\{ \binom{2q+1}{2k+1} y^{2k+1} x^{2q-2k} \right. \\
 &\quad \left. + \sum_{\nu=N}^q [(-1)^{N+\nu+1} \binom{2q+1}{2\nu+1} \binom{\nu}{k} \binom{\nu-k-1}{N-k-1} y^{2\nu+1} x^{2q-2\nu} \right\} j_N^{2k+1}
 \end{aligned}$$

To simplify the writing, we set

$$\begin{aligned}
 (2.33) \quad (x + j_N y)^p &\equiv \sum_{i=0}^{2N-1} \left[\sum_{l=0}^p A_{ilp}^{(N)} y^l x^{p-l} \right] j_N^i, \\
 j_N (x + j_N y)^p &\equiv \sum_{i=0}^{2N-1} \left[\sum_{l=0}^p B_{ilp}^{(N)} y^l x^{p-l} \right] j_N^i,
 \end{aligned}$$

where the $A_{ilp}^{(N)}$ are given explicitly by (2.30) and the $B_{ilp}^{(N)}$ may be obtained immediately using Theorem 2.6.

3. (Σ, N) -monogenic functions. Differentiation and integration.

A. For each positive integer N , we consider the system of equations $E(\Sigma, N)$

$$\begin{aligned}
 (3.1) \quad \sigma_1(x) a_{2k, x} &= \tau_1(y) a_{2k+1, y}, \\
 \sigma_2(x) a_{2k, y} &= \tau_2(y) [a_{2k-1, x} - a_{2N-1, x} \binom{N}{k}],
 \end{aligned}$$

$(k=0, \dots, N-1)$, where $a_i = a_i(x, y)$ for $i=0, \dots, 2N-1$ and $a_{-1} \equiv 0$.

The system (3.1) may be written in a different way. For $k=0$, the second equation of (3.1) yields

$$\sigma_2(x) a_{0, y} = -\tau_2(y) a_{2N-1, x}.$$

With the aid of this last equation, (3.1) becomes

$$\begin{aligned}
 (3.2) \quad \tau_2(y) a_{2k+1, x} &= \sigma_2(x) [a_{2k+2, y} - a_{0, y} \binom{N}{k+1}], \\
 \tau_1(y) a_{2k+1, y} &= \sigma_1(x) a_{2k, x}, \\
 (k=0, \dots, N-1), \text{ where } a_{2N} &\equiv 0.
 \end{aligned}$$

To simplify the writing, we shall use the condensed notation introduced in 1 and rewrite the last two equations in the form

$$\begin{aligned}
 (3.3) \quad a_{2k, X} &= a_{2k+1, Y}, \\
 a_{2k, Y^*} &= a_{2k-1, X^*} - a_{2N-1, X^*} \binom{N}{k},
 \end{aligned}$$

($k = 0, \dots, N-1$), and

$$(3.4) \quad \begin{aligned} a_{2k+1, X^*} &= a_{2k+2, Y^*} - a_{0, Y^*} \binom{N}{k+1}, \\ a_{2k+1, Y} &= a_{2k, X}, \end{aligned}$$

($k = 0, \dots, N-1$).

By analogy with the theory of functions of a complex variable, we lay down the following definitions:

DEFINITION 3.1. A hypercomplex valued function $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$, where $a_i = a_i(x, y)$ for $i = 0, \dots, 2N-1$, is said to be (Σ, N) -monogenic at a point $z = (x, y)$ if there exists a neighborhood D of z such that the functions a_i belong to class $C^{(1)}$ in D and satisfy the system $E(\Sigma, N)$ in D .

We shall say that $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in the domain D if $f(z)$ is (Σ, N) -monogenic at all points of D . With this understanding, we have the

DEFINITION 3.2. If the function $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in a domain D and all the a_i belong to class $C^{(2)}$ in D , then the function $f^{[1]}(z) = \sum_{i=0}^{2N-1} a_i^{[1]} j_N^i$, where

$$(3.5) \quad \begin{aligned} a_{2k}^{[1]} &= a_{2k, X} = a_{2k+1, Y}, \\ a_{2k+1}^{[1]} &= a_{2k+1, X^*} = a_{2k+2, Y^*} - a_{0, Y^*} \binom{N}{k+1}, \end{aligned}$$

($k = 0, \dots, N-1$), is said to be the (Σ, N) -derivative of $f(z)$ and is denoted by $d_{(\Sigma, N)} f / d_{(\Sigma, N)} z$ or more simply by $d_\Sigma f / d_\Sigma z$ when no confusion as to the value of N might arise.

From (3.5) and Theorem 2.6, we may write

$$\begin{aligned} f^{[1]}(z) &= \sum_{k=0}^{N-1} a_{2k, X} j_N^{2k} + \sum_{k=0}^{N-1} a_{2k+1, X^*} j_N^{2k+1} \\ &= \sum_{k=0}^{N-1} a_{2k+1, Y} j_N^{2k} + \sum_{k=0}^{N-1} [a_{2k+2, Y^*} - a_{0, Y^*} \binom{N}{k+1}] j_N^{2k+1} \\ &= (1/j_N) \left[\sum_{k=0}^{N-1} a_{2k, Y^*} j_N^{2k} + \sum_{k=0}^{N-1} a_{2k+1, Y} j_N^{2k+1} \right]. \end{aligned}$$

The following remarks are immediate consequences of the last two definitions: Any hypercomplex constant $\sum_{i=0}^{2N-1} b_i j_N^i$ is (Σ, N) -monogenic, and

conversely any (Σ, N) -monogenic function whose (Σ, N) -derivative vanishes identically is a constant. Furthermore, if f and g are (Σ, N) -monogenic, and α and β are real constants, then $\alpha f + \beta g$ is (Σ, N) -monogenic, and

$$[\alpha f + \beta g]^{[1]} = \alpha f^{[1]} + \beta g^{[1]}.$$

We also have

THEOREM 3.1. *If the function $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic and possesses a (Σ, N) -derivative $f^{[1]}(z)$ in a domain D , then $f^{[1]}(z)$ is (Σ', N) -monogenic in D .*

Proof. From (3.5), it follows that

$$\begin{aligned} a^{[1]}_{2k, X^*} &= a_{2k+1, Y X^*}, & a^{[1]}_{2k+1, Y} &= a_{2k+1, X^* Y}, \\ a^{[1]}_{2k, Y^*} &= a_{2k, X Y^*}, & a^{[1]}_{2k+1, X} &= a_{2k+2, Y^* X} - a_{0 Y^* X} \binom{N}{k+1}, \end{aligned}$$

($k = 0, \dots, N-1$), and since

$$a^{[1]}_{2N-1, X} = -a_{0, Y^* X},$$

we obtain

$$(3.6) \quad \begin{aligned} a^{[1]}_{2k, X^*} &= a^{[1]}_{2k+1, Y}, \\ a^{[1]}_{2k, Y^*} &= a^{[1]}_{2k-1, X} - a^{[1]}_{2N-1, X} \binom{N}{k}, \end{aligned}$$

($k = 0, \dots, N-1$), which is $E(\Sigma', N)$, by (1.14).

Next, we proceed to the definition of higher (Σ, N) -derivatives of a (Σ, N) -monogenic function. If the function $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in a domain D and all the a_i belong to class $C^{(m)}$, $m \geq 2$, then the higher (Σ, N) -derivatives $f^{[n]}(z) = \sum_{i=0}^{2N-1} a_i^{[n]} j_N^i$ are defined by

$$(3.7) \quad \begin{aligned} f^{[0]}(z) &= f(z), \\ f^{[n]}(z) &= \begin{cases} \frac{d_{(\Sigma, N)} f^{[n-1]}(z)}{d_{(\Sigma, N)} z} & \text{if } n \text{ is odd,} \\ \frac{d_{(\Sigma', N)} f^{[n-1]}(z)}{d_{(\Sigma', N)} z} & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

($n = 1, 2, \dots, m-1$). Obviously, $f^{[2n]}(z)$ is (Σ, N) -monogenic and $f^{[2n+1]}(z)$ is (Σ', N) -monogenic.

B. The (Σ, N) -integral of a continuous function $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ over a rectifiable curve C is defined by

$$(3.8) \quad \int_C f(z) d_{(\Sigma, N)} z = \sum_{i=0}^{2N-1} A_i j_N^i,$$

where

$$\begin{aligned} A_{2k} &= \int_C a_{2k} dX^* + [a_{2k-1} - a_{2N-1} \binom{N}{k}] dY^* \\ (3.9) \quad &= \int_C \sigma_2 a_{2k} dx + \tau_2 [a_{2k-1} - a_{2N-1} \binom{N}{k}] dy, \\ A_{2k+1} &= \int_C a_{2k+1} dX + a_{2k} dY \\ &= \int_C (1/\sigma_1) a_{2k+1} dx + (1/\tau_1) a_{2k} dy, \\ (k &= 0, \dots, N-1). \end{aligned}$$

Introducing the ordinary complex-valued functions

$$\begin{aligned} (3.10) \quad h_{2k}(z) &= \sigma_2 a_{2k} - i\tau_2 [a_{2k-1} - a_{2N-1} \binom{N}{k}], \\ h_{2k+1}(z) &= (1/\tau_1) a_{2k} + i(1/\sigma_1) a_{2k+1}, \end{aligned}$$

($k=0, \dots, N-1$), and recalling the expression for $\int_C g(z) dz$, where $dz = dx + idy$, (the ordinary complex integral over C of the continuous function $g(z) = s + it$), equation (3.8) may be rewritten in the form

$$\begin{aligned} (3.11) \quad \int_C f(z) d_{(\Sigma, N)} z &= \sum_{k=0}^{N-1} \{ \operatorname{Re} [\int_C h_{2k}(z) dz] \} j_N^{2k} \\ &+ \sum_{k=0}^{N-1} \{ \operatorname{Im} [\int_C h_{2k+1}(z) dz] \} j_N^{2k+1}. \end{aligned}$$

From (3.8) and (3.9), it follows that if $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ and $g(z) = \sum_{i=0}^{2N-1} b_i j_N^i$ are continuous on C and α and β are real constants, then

$$\int_C (\alpha f + \beta g) d_{(\Sigma, N)} z = \alpha \int_C f d_{(\Sigma, N)} z + \beta \int_C g d_{(\Sigma, N)} z.$$

Also, over C

$$\int_a^b f d_{(\Sigma, N)} z + \int_b^c f d_{(\Sigma, N)} z = \int_a^c f d_{(\Sigma, N)} z.$$

Let l be the length of C ; A , a constant such that for all points of C , $|x| \leq A$, $|y| \leq A$, and $m_\Sigma(A)$ the non-decreasing function defined by $m_\Sigma(A) = \max_{i=1,2} \max_{l=\pm 1} \max_{|t| \leq A} \{\sigma_i(t)^l, \tau_i(t)^l\}$. The following inequality holds

$$(3.12) \quad \left\| \int_C f d_{(\Sigma, N)} z \right\| \leq (2^N + 2N) m_{\Sigma}(A) \cdot \max_C \|f\| \cdot l.$$

For, from (3.11)

$$\left\| \int_C f(z) d_{(\Sigma, N)} z \right\| \leq \left(\sum_{k=0}^{N-1} \max_C |h_{2k}| + \sum_{k=0}^{N-1} \max_C |h_{2k+1}| \right) l,$$

while from (3.10), letting $M = \max_C \|f\|$ and $m = m_{\Sigma}(A)$

$$|h_{2k+1}| \leq 2mM,$$

($k = 0, \dots, N-1$), and

$$\begin{aligned} |h_0| &\leq 2mM, \\ |h_{2k}| &\leq mM(2 + \binom{N}{k}). \end{aligned}$$

THEOREM 3.2. *If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in a closed simply connected domain D bounded by a rectifiable curve C then*

$$(3.13) \quad \int_C f(z) d_{(\Sigma, N)} z = 0.$$

Proof. This follows immediately upon observing that the line integrals in (3.9) are all zero in view of (3.1) or (3.3).

In view of the independence of the path asserted by Theorem 3.2, we may consider the function

$$F(z) = \int_{z_0}^z f d_{(\Sigma, N)} z = \sum_{i=0}^{2N-1} A_i j_N^i,$$

where $f = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in D . It is easily shown that

THEOREM 3.3. *The (Σ, N) -integral of a (Σ, N) -monogenic function is (Σ', N) -monogenic.*

For, from (3.9)

$$(3.14) \quad \begin{aligned} A_{2k, X^*} &= a_{2k}, & A_{2k+1, Y} &= a_{2k}, \\ A_{2k, Y^*} &= a_{2k-1} - a_{2N-1} \binom{N}{k}, & A_{2k+1, X} &= a_{2k+1}, \end{aligned}$$

($k = 0, \dots, N-1$), and since

$$A_{2N-1, X} = a_{2N-1},$$

we obtain $E(\Sigma', N)$ from (3.14).

From (3.14) and Theorem 3.3, by a simple induction, we may prove the following corollary

COROLLARY 3.3. Let $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ be (Σ, N) -monogenic in a simply connected domain D and define

$$(3.15) \quad F_0(z) = f(z),$$

$$F_n(z) = \begin{cases} \int_a^z F_{n-1}(\xi) d_{(\Sigma', N)} \xi & \text{if } n \text{ is odd,} \\ \int_{a^-}^z F_{n-1}(\xi) d_{(\Sigma', N)} \xi & \text{if } n \text{ is even,} \end{cases}$$

for $n \geq 1$, where a is a fixed point of D . Then $F_n(z) = \sum_{i=0}^{2N-1} A_i^{(n)} j_N^i$ is (Σ, N) -monogenic for n even and (Σ', N) -monogenic for n odd. Furthermore, the functions $A_i^{(n)}$ belong to class $C^{(n)}$ and for $q = 0, 1, \dots$

$$(3.16) \quad \frac{\partial^{2q} A_{2k}^{(2q)}}{(\partial X^* \partial X)^q} = a_{2k}, \quad \frac{\partial^{2q} A_{2k+1}^{(2q)}}{(\partial X \partial X^*)^q} = a_{2k+1},$$

$$\frac{\partial^{2q+1} A_{2k}^{(2q+1)}}{\partial X^* (\partial X \partial X^*)^q} = a_{2k}, \quad \frac{\partial^{2q+1} A_{2k+1}^{(2q+1)}}{\partial X (\partial X^* \partial X)^q} = a_{2k+1},$$

($k = 0, \dots, N-1$).

4. Contraction. In the preceding section, we defined (Σ, N) - and (Σ', N) -differentiation and integration. By repeated application of these processes, we can obtain new solutions of the system $E(\Sigma, N)$ from any given solution of this system. Here, we shall be concerned with a process which enables us to find solutions of the systems $E(\Sigma, M)$, $1 \leq M \leq N-1$, given a solution of the system $E(\Sigma, N)$.

LEMMA 4.1. If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in a domain D then $\phi(z) = \sum_{i=0}^{2N-3} \alpha_i j_{N-1}^i$, where

$$(4.1) \quad \alpha_{2k} = a_{2k} - \binom{N-1}{k} a_{2N-2},$$

$$\alpha_{2k+1} = a_{2k+1} - \binom{N-1}{k} a_{2N-1},$$

($k = 0, \dots, N-2$), is $(\Sigma, N-1)$ -monogenic in D .

Proof. This is easily verified using (4.1) and the binomial identity

$$\binom{N-1}{k-1} + \binom{N-1}{k} = \binom{N}{k}.$$

THEOREM 4.1. *If the function $f_N(z) = \sum_{k=0}^{N-1} a_{2k} j_N^{2k} + \sum_{k=0}^{N-1} a_{2k+1} j_N^{2k+1}$ is (Σ, N) -monogenic in a domain D then for each $i=0, \dots, N-1$ the function*

$$(4.2) \quad \begin{aligned} f_{N-i}(z) &= \sum_{k=0}^{N-1} a_{2k} j_{N-i}^{2k} + \sum_{k=0}^{N-1} a_{2k+1} j_{N-i}^{2k+1} \\ &= \sum_{k=0}^{N-i-1} [a_{2k} + \sum_{\nu=N-i}^{N-1} \{(-1)^{N-i+\nu+1} \binom{\nu}{k} \binom{\nu-k-1}{N-i-k-1} a_{2\nu}\}] j_{N-i}^{2k} \\ &\quad + \sum_{k=0}^{N-i-1} [a_{2k+1} + \sum_{\nu=N-i}^{N-1} \{(-1)^{N-i+\nu+1} \binom{\nu}{k} \binom{\nu-k-1}{N-i-k-1} a_{2\nu+1}\}] j_{N-i}^{2k+1} \end{aligned}$$

is $(\Sigma, N-i)$ -monogenic in D .

A sketch of the proof is as follows. First, using (2.28), we verify that the formal substitution of j_{N-i} for j_N in $f_N(z)$ yields the expression on the extreme right of (4.2). To prove that $f_{N-i}(z)$ is $(\Sigma, N-i)$ -monogenic, we proceed by induction, employing Lemma 4.1 and the binomial identity

$$\begin{aligned} \binom{\nu}{k} \binom{\nu-k-1}{N-i-k-1} + \binom{\nu}{k} \binom{\nu-k-1}{N-i-k-2} \\ = \binom{N-i-1}{k} \binom{\nu}{N-i-1}. \end{aligned}$$

In particular, we observe that taking $i=N-1$ shows that

$$f_1(z) = \sum_{k=0}^{N-1} (-1)^k a_{2k} + j_1 \sum_{k=0}^{N-1} (-1)^k a_{2k+1},$$

is always $(\Sigma, 1)$ -monogenic.

5. The partial differential equations of the $2N$ th order. The main purpose of this section is to show that if $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in a domain D , then the real functions a_{2k} ($k=0, \dots, N-1$) satisfy in D the partial differential equation of the $2N$ -th order

$$\left(\frac{\partial^2}{\partial X^* \partial X} + \frac{\partial^2}{\partial Y \partial Y^*} \right) u = 0,$$

and the real functions a_{2k+1} ($k=0, \dots, N-1$) satisfy in D the partial differential equation of the $2N$ -th order

$$\left(\frac{\partial^2}{\partial X \partial X^*} + \frac{\partial^2}{\partial Y^* \partial Y} \right) u = 0.$$

In addition, it will be shown that all the theorems of Section 3 hold under the sole assumption that the functions a_i belong to class $C^{(1)}$.

LEMMA 5.1. If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in a domain D and the a_i belong to class $C^{(3)}$ in D , then the function $g(z) = \sum_{i=0}^{2N-1} b_i j_N^i$, where

$$b_{2k} = a_{2k+1, X^* Y} = (\tau_1 / \sigma_2) a_{2k+1, xy},$$

$$b_{2k+1} = a_{2k+2, XY^*} - a_{0, XY^*} \left(\frac{N}{k+1} \right) = (\sigma_1 / \tau_2) [a_{2k+2, xy} - a_{0, xy} \left(\frac{N}{k+1} \right)],$$

($k = 0, \dots, N-1$), is also (Σ, N) -monogenic in D .

Proof. It is readily verified, from (3.7) and (3.5), that $g(z) = f^{[2]}(z)$. That $g(z)$ is (Σ, N) -monogenic follows from the remark immediately after equation (3.7).

THEOREM 5.1. If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in a domain D and the a_i belong to class $C^{(2N)}$ in D , then

$$(5.1) \quad \begin{aligned} L^N a_{2k} &= 0, \\ M^N a_{2k+1} &= 0, \end{aligned}$$

($k = 0, \dots, N-1$), in D , where L and M are the linear differential operators defined by

$$(5.2) \quad \begin{aligned} Lu &\equiv (\tau_1 / \sigma_2) [(\sigma_1 u_x / \tau_1)_x + (\sigma_2 u_y / \tau_2)_y] = U_{XX^*} + U_{Y^* Y}, \\ Mu &\equiv (\sigma_1 / \tau_2) [(\tau_2 u_x / \sigma)_x + (\tau_1 u_y / \sigma_1)_y] = U_{X^* X} + U_{Y Y^*}, \end{aligned}$$

Proof. We proceed by induction. $E(\Sigma, 1)$ is the system of equations considered by Bers and Gelbart (1)

$$\begin{array}{ll} \sigma_1 a_{0,x} = \tau_1 a_{1,y} & \text{OR} \quad a_{0,X} = a_{1,Y}, \\ \sigma_2 a_{0,y} = -\tau_2 a_{1,x}, & a_{0,Y^*} = -a_{1,X^*}, \end{array}$$

Assuming that a_1 and a_0 belong to class $C^{(2)}$, and eliminating a_1 and a_0 in turn, we obtain

$$L a_0 = 0,$$

$$M a_1 = 0,$$

and the conclusion of the theorem has been verified for $N = 1$.

Suppose now that the conclusion is true for $N-1 \geq 1$ and let $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ be (Σ, N) -monogenic, where the a_i belong to class $C^{(2N)}$.

We have $E(\Sigma, N)$

$$(5.3) \quad \begin{aligned} a_{2k,X} &= a_{2k+1,Y}, \\ a_{2k,Y^*} &= a_{2k-1,X^*} - a_{2N-1,X^*} \binom{N}{k}, \end{aligned}$$

($k=0, \dots, N-1$). By Lemma 4.1, the function $\phi(z) = \sum_{i=0}^{2N-3} \alpha_i j_{N-1}^i$ is $(\Sigma, N-1)$ -monogenic, where

$$(5.4) \quad \begin{aligned} \alpha_{2k} &= a_{2k} - \binom{N-1}{k} a_{2N-2}, \\ \alpha_{2k+1} &= a_{2k+1} - \binom{N-1}{k} a_{2N-1}, \end{aligned}$$

($k=0, \dots, N-2$). Consequently, by the induction hypothesis, since the α_i belong to the class $C^{(2N)}$ in D

$$(5.5) \quad \begin{aligned} L^{N-1} \alpha_{2k} &= 0, \\ M^{N-1} \alpha_{2k+1} &= 0, \end{aligned}$$

($k=0, \dots, N-2$). Since L and M are linear operators, (5.4) and (5.5) imply

$$(5.6) \quad \begin{aligned} L^{N-1} a_{2k} &= \binom{N-1}{k} L^{N-1} a_{2N-2}, \\ M^{N-1} a_{2k+1} &= \binom{N-1}{k} M^{N-1} a_{2N-1}, \end{aligned}$$

($k=0, \dots, N-2$). It is obvious then that to verify (5.1) it suffices to show that $L^N a_{2N-2}$ and $M^N a_{2N-1}$ are both zero.

From (5.3), for $k=0$,

$$(5.7) \quad \begin{aligned} a_{0,X} &= a_{1,Y}, \\ a_{0,Y^*} &= -a_{2N-1,X^*}, \end{aligned}$$

Using the definition of L , (5.2) and (5.4), equation (5.7) yields

$$(5.8) \quad La_0 = \alpha_{1,X^*Y},$$

since

$$\begin{aligned} La_0 &= a_{0,XX^*} + a_{0,Y^*Y} \\ &= a_{1,YX^*} - a_{2N-1,X^*Y} = [a_1 - \binom{N-1}{0} a_{2N-1}]_{X^*Y}. \end{aligned}$$

From (3.4) for $k = 0$

$$(5.9) \quad \begin{aligned} a_{1,X^*} &= a_{2,Y^*} - a_{0,Y^*}(N-1), \\ a_{1,Y} &= a_{0,X}. \end{aligned}$$

Using the definition of M , (5.2), and (5.4), equation (5.9) yields

$$(5.10) \quad Ma_1 = \alpha_{2,XY^*} - \binom{N-1}{1} \alpha_{0,XY^*},$$

since

$$\begin{aligned} Ma_1 &= a_{1,X^*X} + a_{1,YY^*} \\ &= a_{2,Y^*X} - a_{0,Y^*X}(N-1) + a_{0,XY^*} \\ &= [a_2 - \binom{N-1}{1} a_0]_{XY^*}, \end{aligned}$$

and from (5.4)

$$\begin{aligned} \alpha_2 - \binom{N-1}{1} \alpha_0 &= [a_2 - \binom{N-1}{1} a_{2N-2}] - \binom{N-1}{1} [a_0 \\ &\quad - \binom{N-1}{0} a_{2N-2}] = a_2 - \binom{N-1}{1} a_0. \end{aligned}$$

On the other hand, since $N \geq 2$, we may apply Lemma 5.1 to $\phi(z)$ to prove that the function $\phi^{[2]}(z) = \sum_{i=0}^{2N-3} \alpha_i^{[2]} j_{N-1}^i$ is $(\Sigma, N-1)$ -monogenic, where

$$(5.11) \quad \begin{aligned} \alpha_{2k}^{[2]} &= \alpha_{2k+1, X^*Y}, \\ \alpha_{2k+1}^{[2]} &= \alpha_{2k+2, XY^*} - \alpha_{0, XY^*} \binom{N-1}{k+1}, \end{aligned}$$

($k = 0, \dots, N-2$). Consequently, by the induction hypothesis, inasmuch as the α_i belong to class $O^{(2N-2)}$, we have

$$(5.12) \quad \begin{aligned} L^{N-1} \alpha_{2k}^{[2]} &= 0, \\ M^{N-1} \alpha_{2k+1}^{[2]} &= 0, \\ (k &= 0, \dots, N-2). \end{aligned}$$

But a comparison of (5.8), and (5.10) with (5.11) gives

$$(5.13) \quad \begin{aligned} La_0 &= \alpha_0^{[2]}, \\ Ma_1 &= \alpha_1^{[2]}, \end{aligned}$$

and, in view of (5.12), (5.13) yields

$$(5.14) \quad \begin{aligned} L^N a_0 &= 0, \\ M^N a_1 &= 0. \end{aligned}$$

The desired conclusion, (5.1), follows from (5.6) and (5.14).

It is now possible to remove most of the differentiability assumptions made on the a_i in the hypotheses of several theorems. It will be recalled that $\sigma_1, \sigma_2, \tau_1$, and τ_2 are positive analytic functions of their respective real variables. Hence, the operators L and M are of elliptic type with analytic coefficients and any function u of class $C^{(2N)}$ satisfying either $L^N u = 0$ or $M^N u = 0$ for a fixed $N = 1, 2, \dots$, is necessarily analytic. Let $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ be (Σ, N) -monogenic in a simply connected domain D and consider the function

$$F_{2N}(z) = \sum_{i=0}^{2N-1} A_i^{(2N)} j_N^i,$$

(the functions $F_n(z)$ were defined in (3.15)). From Corollary 3.3, it follows that $F_{2N}(z)$ is (Σ, N) -monogenic and that the $A_i^{(2N)}$ belong to class $C^{(2N)}$. Hence, from Theorem 5.1,

$$L^N A_{2k}^{(2N)} = 0,$$

$$M^N A_{2k+1}^{(2N)} = 0,$$

($k = 0, \dots, N-1$), and consequently the $A_i^{(2N)}$ are analytic. But from (3.16)

$$\frac{\partial^{2N} A_{2k}^{(2N)}}{(\partial X^* \partial X)^N} = a_{2k}, \quad \frac{\partial^{2N} A_{2k+1}^{(2N)}}{(\partial X \partial X^*)^N} = a_{2k+1},$$

($k = 0, \dots, N-1$), which shows that $a_i(x, y)$ is a linear combination, with analytic coefficients, of $A_i^{(2N)}(x, y)$ and its partial derivatives with respect to x up to the $2N$ -th order. Thus the functions $a_i(x, y)$ are analytic functions of x and y and we have the

THEOREM 5.2. *If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in a domain D , then the functions $a_i(x, y)$ are analytic functions of x and y in D .*

Using this result, we see that unlimited successive (Σ, N) - and (Σ', N) -differentiations of a (Σ, N) -monogenic function are possible.

Theorem 5.1 may now be restated in the form

THEOREM 5.3. *If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in a domain D , then*

$$L^N a_{2k} = 0,$$

$$M^N a_{2k+1} = 0,$$

($k = 0, \dots, N-1$), in D .

We remark that (Σ, N) -integration and (Σ', N) -differentiation, as well as (Σ, N) -differentiation and (Σ', N) -integration, are inverse processes. That is, if $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic, then

$$(5.15) \quad \frac{d_{(\Sigma', N)}}{d_{(\Sigma', N)} z} \int_{z_0}^z f(\xi) d_{(\Sigma, N)} \xi = f(z),$$

and

$$(5.16) \quad \int_{z_0}^z f^{[1]}(\xi) d_{(\Sigma', N)} \xi = f(z) - f(z_0),$$

Next, we prove the analogue of Morera's theorem.

THEOREM 5.4. *If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is continuous in a simply connected domain D and for every simple closed rectifiable curve C in D , $\int_C f(z) d_{(\Sigma, N)} z = 0$, then $f(z)$ is (Σ, N) -monogenic in D .*

Proof. Consider the function

$$F(z) = \int_{z_0}^z f d_{(\Sigma, N)} z = \sum_{i=0}^{2N-1} A_i j_N^i.$$

By (3.14), as in the proof of Theorem 3.3, we see that $f(z)$ is (Σ', N) -monogenic. By (3.14) and (3.5), $\frac{d_{(\Sigma', N)} F}{d_{(\Sigma', N)} z} = f(z)$.

THEOREM 5.5. *If a series of (Σ, N) -monogenic functions $\sum_{n=0}^{\infty} f_n(z)$, where $f_n(z) = \sum_{i=0}^{\infty} a_{ni} j_N^i$, converges uniformly to $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ in a domain D , then $f(z)$ is (Σ, N) -monogenic in D .*

6. The associated matrices. The system $E(\Sigma, N)$ has

$$\Sigma = \begin{vmatrix} \sigma_1 & \tau_1 \\ \sigma_2 & \tau_2 \end{vmatrix}.$$

as the matrix of its coefficients. Together with $E(\Sigma, N)$, we shall be led to consider systems whose coefficient matrices are

$$(6.1) \quad \Sigma' = \begin{vmatrix} 1/\sigma_2 & \tau_1 \\ 1/\sigma_1 & \tau_2 \end{vmatrix}, \quad \Sigma_{\prime} = \begin{vmatrix} \sigma_1 & 1/\tau_2 \\ \sigma_2 & 1/\tau_1 \end{vmatrix}, \quad \Sigma_{\prime\prime} = \begin{vmatrix} 1/\sigma_2 & 1/\tau_2 \\ 1/\sigma_1 & 1/\tau_1 \end{vmatrix}.$$

Obviously,

$$\Sigma'' = \Sigma_{\prime\prime} = \Sigma_{\prime\prime\prime} = \Sigma.$$

It has already been shown that the (Σ, N) -integral and the (Σ, N) -derivative of a (Σ, N) -monogenic function are both (Σ', N) -monogenic functions. We now prove two theorems concerning the relations between (Σ, N) -, (Σ', N) -, (Σ, N) -, and (Σ', N) -monogenic functions.

THEOREM 6.1. *If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic, then $g(z) \equiv j_N f(z)$ is (Σ', N) -monogenic.*

Letting $g(z) = \sum_{i=0}^{2N-1} b_i j_N^i$ and using Theorem 2.6, the result is immediate.

COROLLARY 6.1. *If $f(z)$ is (Σ, N) -monogenic, then $j_N^{2k} f(z)$ is (Σ, N) -monogenic and $j_N^{2k+1} f(z)$ is (Σ', N) -monogenic, where k is any integer.*

The proof follows for positive k by repeated application of Theorem 6.1, and then for negative k by using Theorem 2.5.

THEOREM 6.2. *If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic, then*

$$(6.3) \quad \frac{d_{(\Sigma', N)} [j_N f]}{d_{(\Sigma', N)} z} = j_N \frac{d_{(\Sigma, N)} f}{d_{(\Sigma, N)} z},$$

and

$$(6.4) \quad \int_{z_0}^z j_N f d_{(\Sigma', N)} z = j_N \int_{z_0}^z f d_{(\Sigma, N)} z.$$

Proof. (6.3) is a consequence of the definition of the (Σ, N) -derivative, equation (3.5), and Theorem 6.1. As regards (6.4), both sides of the proposed equality are (Σ, N) -monogenic functions. In addition, both functions vanish at z_0 and, by (6.3), the (Σ, N) -derivative of their difference vanishes identically.

COROLLARY 6.2. *If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic, then*

$$(6.5) \quad \frac{d_{(\Sigma, N)} [j_N^{2k} f]}{d_{(\Sigma, N)} z} = j_N^{2k} \frac{d_{(\Sigma, N)} f}{d_{(\Sigma, N)} z},$$

$$(6.6) \quad \int_{z_0}^z j_N^{2k} f d_{(\Sigma, N)} z = j_N^{2k} \int_{z_0}^z f d_{(\Sigma, N)} z,$$

$$(6.7) \quad \frac{d_{(\Sigma', N)} [j_N^{2k+1} f]}{d_{(\Sigma', N)} z} = j_N^{2k+1} \frac{d_{(\Sigma, N)} f}{d_{(\Sigma, N)} z},$$

$$(6.8) \quad \int_{z_0}^z j_N^{2k+1} f d_{(\Sigma', N)} z = j_N^{2k+1} \int_{z_0}^z f d_{(\Sigma, N)} z,$$

where k is any integer.

7. The formal powers.

A. Any hypercomplex constant is a (Σ, N) -monogenic function; hence, we may construct (Σ, N) -monogenic functions by successive (Σ, N) - and (Σ', N) -integrations of a constant. Accordingly, we define [see Bers and Gelbart (1), Section 4]

$$\begin{aligned} a \cdot Z^{(0)}(N; z_0; z) &= a \cdot \tilde{Z}^{(0)}(N; z_0; z) = a, \\ (7.1) \quad a \cdot Z^{(n)}(N; z_0; z) &= n \int_{z_0}^z (a \cdot \tilde{Z}^{(n-1)}(N; z_0; z)) d_{(\Sigma', N)} z, \\ a \cdot \tilde{Z}^{(n)}(N; z_0; z) &= n \int_{z_0}^z (a \cdot Z^{(n-1)}(N; z_0; z)) d_{(\Sigma, N)} z, \end{aligned}$$

for $n \geq 1$, where $a = \sum_{i=0}^{2N-1} \alpha_i j_N^i$ is a hypercomplex constant and the point z_0 is fixed. In this formula $a \cdot Z$ is *one* symbol and does not denote ordinary multiplication of a by Z , since no meaning has been attached to Z by itself as yet. In addition, we have

$$(7.2) \quad a \cdot Z^{(n)}(N; z_0; z_0) = a \cdot \tilde{Z}^{(n)}(N; z_0; z_0) = 0$$

and

$$\begin{aligned} (7.3) \quad \frac{d_{(\Sigma, N)}(a \cdot Z^{(n)}(N; z_0; z))}{d_{(\Sigma, N)} z} &= na \cdot \tilde{Z}^{(n-1)}(N; z_0; z), \\ \frac{d_{(\Sigma', N)}(a \cdot \tilde{Z}^{(n)}(N; z_0; z))}{d_{(\Sigma', N)} z} &= na \cdot Z^{(n-1)}(N; z_0; z). \end{aligned}$$

In this section, we shall keep N fixed. To simplify the writing, we set

$$1 \cdot Z^{(n)}(N; z_0; z) \equiv Z^{(n)}(z_0; z), \quad a \cdot Z^{(n)}(N; z_0; z) \equiv a \cdot Z^{(n)}(z_0; z),$$

and similarly for the $\tilde{Z}^{(n)}$'s. Also, we shall write Σ -monogenic instead of (Σ, N) -monogenic, and use j in place of j_N .

At first glance, the iterated Σ - and Σ' -integration of the $2N$ constants $1, j, j^2, \dots, j^{2N-1}$ might be expected to yield $2N$ different functions as the result of each integration operation. However, each time, essentially only two new functions appear, because

$$\begin{aligned} (7.4) \quad j^{2k} \cdot Z^{(n)}(z_0; z) &= j^{2k} Z^{(n)}(z_0; z), \\ j^{2k+1} \cdot Z^{(n)}(z_0; z) &= j^{2k} [j \cdot Z^{(n)}(z_0; z)], \end{aligned}$$

and

$$\begin{aligned} (7.5) \quad j^{2k} \cdot \tilde{Z}^{(n)}(z_0; z) &= j^{2k} \tilde{Z}^{(n)}(z_0; z), \\ j^{2k+1} \cdot \tilde{Z}^{(n)}(z_0; z) &= j^{2k} [j \cdot \tilde{Z}^{(n)}(z_0; z)] \end{aligned}$$

where $k = 0, 1, 2, \dots$. Obviously, (7.4) and (7.5) hold for $n = 0$. And if they hold for $n - 1$, then they also hold for n , since in view of (7.1) and (6.6)

$$\begin{aligned} j^{2k} \cdot Z^{(n)}(z_0; z) &= n \int_{z_0}^z (j^{2k} \cdot \tilde{Z}^{(n-1)}(z_0; z)) d_{\Sigma'} z \\ &= n \int_{z_0}^z j^{2k} \tilde{Z}^{(n-1)}(z_0; z) d_{\Sigma'} z \\ &= j^{2k} n \int_{z_0}^z \tilde{Z}^{(n-1)}(z_0; z) d_{\Sigma'} z \\ &= j^{2k} Z^{(n)}(z_0; z), \end{aligned}$$

which verifies the first equation of (7.4). A similar reasoning yields the second equation of (7.4). The two equations of (7.5) may be derived analogously, or obtained at once by noticing that if the formal powers formed with respect to the matrix Σ' are denoted by $a \cdot Z'^{(n)}$, etc., we have

$$\begin{aligned} (7.6) \quad a \cdot Z'^{(n)}(z_0; z) &= a \cdot \tilde{Z}^{(n)}(z_0; z), \\ a \cdot \tilde{Z}'^{(n)}(z_0; z) &= a \cdot Z^{(n)}(z_0; z), \end{aligned}$$

and that (7.4) written with respect to Σ' is (7.5).

Clearly, for $a = \sum_{i=0}^{2N-1} \alpha_i j^i$, the α_i being real constants,

$$(7.7) \quad a \cdot Z^{(n)}(z_0; z) = \sum_{i=0}^{2N-1} \alpha_i [j^i \cdot Z^{(n)}(z_0; z)],$$

and similarly for $a \cdot \tilde{Z}^{(n)}(z_0; z)$.

With the aid of (7.4), we obtain, from (7.7), the formula

$$\begin{aligned} (7.8) \quad \left[\sum_{i=0}^{2N-1} \alpha_i j^i \right] \cdot Z^{(n)}(z_0; z) &= \left[\sum_{k=0}^{N-1} \alpha_{2k} j^{2k} \right] Z^{(n)}(z_0; z) \\ &\quad + \left[\sum_{k=0}^{N-1} \alpha_{2k+1} j^{2k+1} \right] [j \cdot Z^{(n)}(z_0; z)], \end{aligned}$$

which gives $a \cdot Z^{(n)}$ explicitly, once $Z^{(n)}$ and $j \cdot Z^{(n)}$ are known.

We call $a \cdot Z^{(n)}$ the "formal product" of " a " and the "formal power" $Z^{(n)}$.

B. If $f(z) = \sum_{i=0}^{2N-1} a_i j^i$, $a_i = a_i(x, y)$, is Σ -monogenic, then the a_i satisfy $E(\Sigma, N)$. Consequently, the formal powers $a \cdot Z^{(n)}(z_0; z)$ furnish us an infinite number of particular solutions of the system $E(\Sigma, N)$. In order to

further identify these particular solutions, and to obtain formulas which are formally equivalent to the binomial formula for $a(z - z_0)^n$, we use the following real functions (1): [See (1.10)]

$$\begin{aligned}
 X^{(0)}(x_0; x) &= X^{*(0)}(x_0; x) = Y^{(0)}(y_0; y) = Y^{*(0)}(y_0; y) \equiv 1 \\
 X^{(n)}(x_0; x) &= \begin{cases} n \int_{x_0}^x X^{(n-1)} dX & \text{if } n \text{ is odd,} \\ n \int_{x_0}^x X^{(n-1)} dX^* & \text{if } n \text{ is even,} \end{cases} \\
 X^{*(n)}(x_0; x) &= \begin{cases} n \int_{x_0}^x X^{*(n-1)} dX^* & \text{if } n \text{ is odd,} \\ n \int_{x_0}^x X^{*(n-1)} dX & \text{if } n \text{ is even,} \end{cases} \\
 Y^{(n)}(y_0; y) &= \begin{cases} n \int_{y_0}^y Y^{(n-1)} dY & \text{if } n \text{ is odd,} \\ n \int_{y_0}^y Y^{(n-1)} dY^* & \text{if } n \text{ is even,} \end{cases} \\
 Y^{*(n)}(y_0; y) &= \begin{cases} n \int_{y_0}^y Y^{*(n-1)} dY^* & \text{if } n \text{ is odd,} \\ n \int_{y_0}^y Y^{*(n-1)} dY & \text{if } n \text{ is even.} \end{cases}
 \end{aligned}
 \tag{7.9}$$

In other words:

$$\begin{aligned}
 X^{(n)}(x_0; x) &= n! \int_{x_0}^x 1/\sigma_1 \int_{x_0}^{\sigma_1} \sigma_2 \cdots \int_{x_0}^{\sigma_{n-1}} \int_{x_0}^{\sigma_n} dx^n / \sigma_1 \quad (n \text{ integrals, } n \text{ odd}) \\
 X^{(n)}(x_0; x) &= n! \int_{x_0}^x \sigma_2 \int_{x_0}^{\sigma_1} 1/\sigma_1 \cdots \int_{x_0}^{\sigma_{n-1}} \int_{x_0}^{\sigma_n} dx^n / \sigma_1 \quad (n \text{ integrals, } n \text{ even})
 \end{aligned}$$

and similarly for $X^{*(n)}$, $Y^{(n)}$, $Y^{*(n)}$. If $x_0 = 0$ (or $y_0 = 0$), we write simply $X^{(n)}(x)$, $X^{*(n)}(x)$, (or $Y^{(n)}(y)$, $Y^{*(n)}(y)$). It is to be noticed that the definition of these functions is independent of N .

Making use of the functions introduced in (7.9), we shall be able to obtain an analogue of the binomial formula for $(x + j_N y)^p$, by replacing the powers of x and y in (2.31) and (2.32) by the appropriate functions of (7.9). We have, for $q = 0, 1, 2, \dots$, in the notation of (2.33),

$$(7.10) \quad Z^{(2q)}(z_0; z) = \sum_{i=0}^{2N-1} \left[\sum_{l=0}^{2q} A_{i,l,2q}^{(N)} Y^{(l)}(y_0; y) X^{*(2q-l)}(x_0; x) \right] j^i$$

$$(7.11) \quad Z^{(2q+1)}(z_0; z) = \sum_{i=0}^{2N-1} \left[\sum_{l=0}^{2q+1} A_{i,l,2q+1}^{(N)} Y^{(l)}(y_0; y) X^{(2q+1-l)}(x_0; x) \right] j^i$$

$$(7.12) \quad j \cdot Z^{(2q)}(z_0; z) = \sum_{i=0}^{2N-1} \left[\sum_{l=0}^{2q} B_{i,l,2q}^{(N)} Y^{*(l)}(y_0; y) X^{(2q-l)}(x_0; x) \right] j^i$$

$$(7.13) \quad j \cdot Z^{(2q+1)}(z_0; z) = \sum_{i=0}^{2N-1} \left[\sum_{l=0}^{2q-1} B_{i,l,2q+1}^{(N)} Y^{*(l)}(y_0; y) X^{*(2q+1-l)}(x_0; x) \right] j^i.$$

Equation (7.10) may be verified by induction, employing the binomial identity (2.29). (7.11) is then obtained by Σ - and Σ' -differentiating

$\frac{1}{(2q+2)(2q+1)} Z^{(2q+2)}(z_0; z)$. (7.12) and (7.13) follow immediately, after observing that if we introduce the formal powers $a \cdot Z',^{(n)}$, etc., which are formed with respect to the matrix Σ' , we have

$$j \cdot Z^{(n)}(z_0; z) = j Z',^{(n)}(z_0; z).$$

C. It is possible to obtain a simple inequality for the absolute value of the formal powers. We have

$$(7.14) \quad \|\alpha \cdot Z^{(n)}(z_0; z)\| \leq 2 \|\alpha\| \Gamma_N [2^{\frac{1}{2}} \Gamma_N m_{\Sigma} (|z_0| + |z|) |z - z_0|]^n,$$

where $\alpha = \sum_{i=0}^{2N-1} \alpha_i j^i$ is a hypercomplex constant, Γ_N is the constant occurring in (2.26), and $|z_0|^2 = x_0^2 + y_0^2$, $|z|^2 = x^2 + y^2$.

First, we obtain inequalities for $Z^{(n)}(z_0; z)$ and for $j \cdot Z^{(n)}(z_0; z)$. Using the non-decreasing function $m_{\Sigma}(A)$, defined by

$$m_{\Sigma}(A) = \max_{i=1,2} \max_{l=1} \max_{|t| \leq A} \{\sigma_i(t)^l, \tau_i(t)^l\},$$

we have that [see (1), page 75]

$$(7.15) \quad m_{\Sigma}^{-n} (|x_0| + |x|) \leq \frac{X^{(n)}(x_0; x)}{(x - x_0)^n} \leq m_{\Sigma}^n (|x_0| + |x|),$$

with similar inequalities for X^* , Y and Y^* . We also know [see (2.31) and (2.32)] that

$$(7.16) \quad Z^{(n)}(z_0; z) = \begin{cases} \sum_{i=0}^n \binom{n}{i} X^{(n-i)} Y^{(i)} j^i & \text{if } n \text{ is odd,} \\ \sum_{i=0}^n \binom{n}{i} X^{*(n-i)} Y^{(i)} j^i & \text{if } n \text{ is even,} \end{cases}$$

$$j \cdot Z^{(n)}(z_0; z) = \begin{cases} j \sum_{i=0}^n \binom{n}{i} X^{*(n-i)} Y^{*(i)} j^i & \text{if } n \text{ is odd,} \\ j \sum_{i=0}^n \binom{n}{i} X^{(n-i)} Y^{*(i)} j^i & \text{if } n \text{ is even.} \end{cases}$$

Consequently,

$$(\gamma.17) \quad \left\{ \begin{array}{l} \| Z^{(n)}(z_0; z) \| \\ \| j \cdot Z^{(n)}(z_0; z) \| \end{array} \right\} \leq 2^{n/2} \Gamma_N^n m_{\Sigma}^n (|z_0| + |z|) |z - z_0|^n,$$

since by (2.26), (2.24), (2.25), plus (2.27) with $\alpha = j$, we have

$$\begin{aligned} \| j \cdot Z^{(n)}(z_0; z) \| &\leq \begin{cases} \Gamma_N \| j \| \left\| \sum_{i=0}^n \binom{n}{i} X^{*(n-i)} Y^{*(i)} j^i \right\| \\ \Gamma_N \| j \| \left\| \sum_{i=0}^n \binom{n}{i} X^{(n-i)} Y^{*(i)} j^i \right\| \end{cases} \\ &\leq \Gamma_N \sum_{i=0}^n \binom{n}{i} m_{\Sigma}^n (|z_0| + |z|) |x - x_0|^{n-i} |y - y_0|^i \Gamma_N^{i-1} \\ &\leq m_{\Sigma}^n (|z_0| + |z|) \Gamma_N^n [|x - x_0| + |y - y_0|]^n \\ &\leq 2^{n/2} \Gamma_N^n m_{\Sigma}^n (|z_0| + |z|) |z - z_0|^n. \end{aligned}$$

The first inequality of (7.17) may be verified similarly.

But from (7.8) and (2.27)

$$\begin{aligned} (\gamma.18) \quad \| \alpha \cdot Z^{(n)}(z_0; z) \| &\leq \Gamma_N \left\| \sum_{k=0}^{N-1} \alpha_{2k} j^{2k} \right\| \| Z^{(n)}(z_0; z) \| \\ &+ \Gamma_N \left\| \sum_{k=0}^{N-1} \alpha_{2k+1} j^{2k} \right\| \| j \cdot Z^{(n)}(z_0; z) \| \\ &\leq \Gamma_N \| \alpha \| [\| Z^{(n)}(z_0; z) \| + \| j \cdot Z^{(n)}(z_0; z) \|], \end{aligned}$$

Equation (7.14) then follows from (7.17) and (7.18).

8. Null functions. There are certain (Σ, N) -monogenic functions $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ such that one or more of the real functions a_i vanish identically. This section will be devoted to the determination of all such "null functions."

THEOREM 8.1. *Let D be a domain containing the point z_0 . If $f(z) = \sum_{i=0}^q a_i j_N^i$ where $0 \leq q \leq 2N-2$, is (Σ, N) -monogenic in D , then*

$$(8.1) \quad f(z) = \sum_{i=0}^q \alpha_i \cdot Z^{(i)}(z_0; z),$$

where

$$(8.2) \quad \alpha_i = \sum_{l=0}^{q-i} \alpha_{i+l} j_N^l,$$

$(i=0, \dots, q)$ are hypercomplex constants. (Conversely, any function defined by (8.1) and (8.2) is of the form $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$, where $a_{q+1} = a_{q+2} = \dots = a_{2N-1} \equiv 0$).

Proof. Given any matrix Σ , the conclusion is true for $q = 0$. For then $f(z)$ is merely a real constant. Suppose that for any matrix Σ the assertion is true for q and let

$$(8.3) \quad f(z) = \sum_{i=0}^{2N-1} a_i j_N^i, \quad a_{q+2} = a_{q+3} = \cdots = a_{2N-1} \equiv 0,$$

be (Σ, N) -monogenic in D . Then

$$(8.4) \quad \begin{aligned} a_{2k, X} &= a_{2k+1, Y}, \\ a_{2k, Y^*} &= a_{2k-1, X^*}, \end{aligned}$$

($k = 0, \cdots, N-1$). If $q+1 = 2k$ for some k , then, in view of the first equation of (8.4),

$$a_{q+1, X} = a_{q+2, Y} = 0.$$

If $q+1 = 2k+1$ for some k , then, in view of the second equation of (8.4),

$$0 = a_{q+2, Y^*} = a_{q+1, X^*}.$$

From Definition 3.2, we see that

$$(8.5) \quad \frac{d_{(\Sigma, N)} f}{d_{(\Sigma, N)} z} = \sum_{i=0}^q b_i j_N^i.$$

The function $\frac{d_{(\Sigma, N)} f}{d_{(\Sigma, N)} z}$ is (Σ', N) -monogenic; hence, by virtue of the induction hypothesis, (8.5) and (8.1) give

$$(8.6) \quad \frac{d_{(\Sigma, N)} f}{d_{(\Sigma, N)} z} = \sum_{i=0}^q \beta_i \cdot \tilde{Z}^{(i)}(z_0; z),$$

where

$$(8.7) \quad \beta_i = \sum_{l=0}^{q-i} \beta_{il} j_N^l,$$

($i = 0, \cdots, q$) are hypercomplex constants.

Equation (8.6) may be rewritten as follows:

$$(8.8) \quad \frac{d_{(\Sigma, N)} f}{d_{(\Sigma, N)} z} = \sum_{i=1}^{q+1} \beta_{i-1} \cdot \tilde{Z}^{(i-1)}(z_0; z) = \sum_{i=1}^{q+1} i \alpha_i \cdot \tilde{Z}^{(i-1)}(z_0; z),$$

by defining

$$(8.9) \quad \alpha_i = \frac{\beta_{i-1}}{i},$$

($i = 1, \cdots, q+1$). Setting

$$x_{i-1} = \frac{\beta_{i-1, l}}{i},$$

for $(i = 1, \dots, q+1)$, $(l = 0, \dots, q-i)$, gives

$$(8.10) \quad \alpha_i = \sum_{l=0}^{q+1-i} \alpha_{il} j_N^l, \\ (i = 1, \dots, q+1).$$

The function $f(z)$ of (8.3) is obtained by (Σ', N) -integrating (8.8). [This (Σ', N) -integration is possible whether D is simply connected or not, since the function given by (8.8) is (Σ', N) -monogenic at all points of the plane.] The result is

$$(8.11) \quad f(z) = \sum_{i=1}^{q+1} \alpha_i \cdot Z^{(i)}(z_0; z) + \alpha_0 = \sum_{i=0}^{q+1} \alpha_i \cdot Z^{(i)}(z_0; z),$$

where

$$(8.12) \quad \alpha_0 = \sum_{l=0}^{q+1} \alpha_{0l} j_N^l,$$

is a hypercomplex constant of a nature determined by the condition (8.3) on the functions a_i . The last three equations serve to verify the conclusion for $q+1$.

LEMMA 8.1. *For each $n \geq 1$, let E_n be the set of $2n$ functions*

$$(8.13) \quad F_{2k} = X^{(2k)}, \\ F_{2k+1} = X^{*(2k+1)},$$

$(k = 0, \dots, n-1)$. E_n is a linearly independent set of functions on any interval $a < x < b$.

The proof is by induction.

THEOREM 8.2a. *Let D be a domain and z_0 be a point of D . If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in D and $a_{2k} \equiv 0$, for some fixed k , where $0 \leq k \leq N-1$, then*

$$(8.14) \quad f(z) = 1/K_{N,k} \sum_{i=0}^{2N-2} \alpha_i \cdot Z^{(i)}(z_0; z),$$

where

$$(8.15) \quad \alpha_i = \sum_{l=1}^{2N-1-i} \alpha_{il} j_N^l,$$

$(i = 0, 1, \dots, 2N-2)$ are hypercomplex constants and the $K_{N,k}$ are given by Theorem 2.7. [Conversely, any function defined by (8.14) and (8.15) is of the form $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$, where $a_{2k} \equiv 0$.]

Proof. The function

$$(8.16) \quad F(z) = \sum_{i=0}^{2N-1} A_i j_N^i = K_{N,k} f(z),$$

is (Σ, N) -monogenic and $A_0 = a_{2k} = 0$. In view of the second equation of $E(\Sigma, N)$, $0 = A_{0,Y^*} = A_{2N-1,X^*}$. Hence, the (Σ', N) -monogenic function $\frac{d_{(\Sigma,N)} F}{d_{(\Sigma,N)} z}$ has the form

$$(8.17) \quad \frac{d_{(\Sigma,N)} F}{d_{(\Sigma,N)} z} = \sum_{i=0}^{2N-2} A_i^{[1]} j_N^i.$$

By Theorem 8.1, we have

$$(8.18) \quad \frac{d_{(\Sigma,N)} F}{d_{(\Sigma,N)} z} = \sum_{i=0}^{2N-2} \beta_i \cdot \tilde{Z}^{(i)}(z_0; z),$$

where

$$(8.19) \quad \beta_i = \sum_{l=0}^{2N-2-i} \beta_{il} j_N^l,$$

($i = 0, \dots, 2N-2$) are hypercomplex constants.

In addition, $A_0^{[1]} = A_{0,X} = 0$, since $A_0 = 0$. Recalling (7.8), and by virtue of (2.31) and (2.32), for $0 \leq q \leq N-1$

$$(8.20) \quad \begin{aligned} \operatorname{Re}[\tilde{Z}^{(2q)}(z_0; z)] &= X^{(2q)}, \\ \operatorname{Re}[j_N \cdot \tilde{Z}^{(2q)}(z_0; z)] &= 0, \end{aligned}$$

and for $0 \leq q \leq N-2$

$$(8.21) \quad \begin{aligned} \operatorname{Re}[\tilde{Z}^{(2q+1)}(z_0; z)] &= X^{*(2q+1)}, \\ \operatorname{Re}[j_N \cdot \tilde{Z}^{(2q+1)}(z_0; z)] &= 0, \end{aligned}$$

we obtain from (8.17) and (8.18), by equating the real part of $\frac{d_{(\Sigma,N)} F}{d_{(\Sigma,N)} z}$ to zero

$$(8.22) \quad A_0^{[1]} = \sum_{h=0}^{N-1} \beta_{2h,0} X^{(2h)} + \sum_{h=0}^{N-2} \beta_{2h+1,0} X^{*(2h+1)} \equiv 0.$$

By Lemma 8.1, (8.22) implies

$$\beta_{i,0} = 0,$$

($i = 0, 1, 2, \dots, 2N-2$). Consequently, $\beta_{2N-2} = 0$ in (8.19) and equations (8.18) and (8.19) become

$$(8.23) \quad \frac{d_{(\Sigma,N)} F}{d_{(\Sigma,N)} z} = \sum_{i=0}^{2N-3} \beta_i \cdot \tilde{Z}^{(i)}(z_0; z),$$

and

$$(8.24) \quad \beta_i = \sum_{l=1}^{2N-2-i} \beta_{il} j_N^l, \\ (i = 0, \dots, 2N-3).$$

Let

$$(8.25) \quad \alpha_i = \frac{\beta_{i-1}}{i} = 1/i \sum_{l=1}^{2N-2-(i-1)} \beta_{i-1,l} j_N^l,$$

($i = 1, \dots, 2N-2$), and

$$(8.26) \quad \alpha_{il} = \beta_{i-1,l}/i,$$

for $i = 1, \dots, 2N-2$, $l = 1, \dots, 2N-1-i$. Then, from (8.24), (8.25), and (8.26)

$$(8.27) \quad \alpha_i = \sum_{l=1}^{2N-1-i} \alpha_{il} j_N^l.$$

($i = 1, \dots, 2N-2$).

Using (8.25) and (8.27), equation (8.23) may be rewritten thus

$$(8.28) \quad \frac{d_{(\Sigma, N)} F}{d_{(\Sigma, N)} z} = \sum_{i=1}^{2N-2} \beta_{i-1} \cdot \tilde{Z}^{(i-1)}(z_0; z) = \sum_{i=1}^{2N-2} i \alpha_i \cdot \tilde{Z}^{(i-1)}(z_0; z).$$

$F(z)$ is found by (Σ', N) -integrating (8.28).

$$(8.29) \quad F(z) = \sum_{i=1}^{2N-2} \alpha_i \cdot \tilde{Z}^{(i)}(z_0; z) + \alpha_0 = \sum_{i=0}^{2N-2} \alpha_i \cdot Z^{(i)}(z_0; z),$$

where

$$(8.30) \quad \alpha_0 = \sum_{l=1}^{2N-1} \alpha_{0l} j_N^l,$$

is a hypercomplex constant of a nature determined by the condition that $A_0 = 0$ in (8.16). Equations (8.29) and (8.16) yield (8.14), and equations (8.30) and (8.27) yield (8.15).

By a similar reasoning, we obtain

THEOREM 8.2b. *Let D be a domain and let z_0 be a point of D . If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in D and $a_{2k+1} \equiv 0$ for some fixed k , where $0 \leq k \leq N-1$, then*

$$(8.31) \quad f(z) = 1/K_{N,k} \left[\sum_{i=1}^{2N-2} \alpha_i \cdot Z^{(i)}(z_0; z) + \alpha_0 \right],$$

where

$$(8.32) \quad \alpha_i = \sum_{l=2}^{2N-i} \alpha_{il} j_N^l,$$

($i = 1, \dots, 2N - 2$) are hypercomplex constants, $\alpha_0 = \sum_{l=0}^{2N-1} \alpha_{0l} j_N^l$ is a hypercomplex constant such that $\alpha_{01} = 0$, and $K_{N,k}$ is given by Theorem 2.7. (Conversely, any function defined by (8.31) and (8.32) is of the form $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$, where $a_{2k+1} \equiv 0$.)

THEOREM 8.3. If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in a domain D and one of the functions a_i vanishes identically in D , then $f(z)$ is defined, single-valued, and (Σ, N) -monogenic over the whole plane.

Proof. In D , $f(z)$ must coincide with either a function of the form (8.14) or a function of the form (8.31), and has the desired properties. Since the "components" a_i of a (Σ, N) -monogenic function are analytic functions of x and y , they are uniquely determined by their values in any neighborhood.

9. (Σ, N) -monogenic functions with prescribed a_i . If $f(z) = u + iv$ is an analytic function of a complex variable $x + iy$, then u and v , the real and imaginary parts of $f(z)$, satisfy Laplace's equation. The analogue of this proposition for (Σ, N) -monogenic functions was considered in 5. Again, given a harmonic function $u = u(x, y)$ there exist two analytic functions of $x + iy$, one having u as its real part and the other having u as its imaginary part. This section will be concerned with the analogue of this "inverse" problem for (Σ, N) -monogenic functions. (A special case has already been treated in 8.)

To facilitate later computations, we note that

$$(9.1) \quad \begin{aligned} M^N u &= \left(\frac{\partial^2}{\partial X \partial Y^*} + \frac{\partial^2}{\partial Y^* \partial X} \right)^n u = \sum_{k=0}^n \binom{n}{k} \frac{\partial^{2n} u}{(\partial X \partial X^*)^k (\partial Y^* \partial Y)^{n-k}}, \\ L^N u &= \left(\frac{\partial^2}{\partial X^* \partial X} + \frac{\partial^2}{\partial Y \partial Y^*} \right)^n u = \sum_{k=0}^n \binom{n}{k} \frac{\partial^{2n} u}{(\partial X^* \partial X)^k (\partial Y \partial Y^*)^{n-k}}. \end{aligned}$$

We treat first the case when the real part a_0 is prescribed.

LEMMA 9.1. Suppose that $u = u(x, y)$ of class $C^{(2N)}$ satisfies the equation $L^N u = 0$ in a domain D . Then, there exists a function $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ which is (Σ, N) -monogenic in D and such that $a_0 = u$.

Proof. We shall prove our result first when D is a rectangle with sides parallel to the axes. For convenience, we suppose that the origin is an interior

point of D . The proof is by induction. The conclusion is true for $N = 1$ for any matrix Σ . In what follows, we suppose that the conclusion is true for $N - 1$ for any matrix Σ .

It follows that given a function $v = v(x, y)$ of class $C^{(2N-2)}$ satisfying $M^{N-1}v = 0$ in D , there exists a function $g(z) = \sum_{i=0}^{2N-3} b_i j_{N-1}^i$ which is $(\Sigma, N - 1)$ -monogenic in D and such that $b_1 = v$. Because for the matrix $\Sigma',$ of (6.1), we have

$$L', = v_{X^*X} + v_{Y^*Y} = Mv$$

and consequently $L',^{N-1}v = M^{N-1}v = 0$ for the given function v . Hence by the induction hypothesis, there exists a function $f(z) = \sum_{i=0}^{2N-3} a_i j_{N-1}^i$ which is $(\Sigma', N - 1)$ -monogenic in D and such that $a_0 = v$. From Theorem 6.1, it follows that $\sum_{i=0}^{2N-1} b_i j_{N-1}^i = g(z) = j_{N-1}f(z)$ is $(\Sigma, N - 1)$ -monogenic, and from Theorem 2.6 that $b_1 = a_0 = v$.

Now, let $u = u(x, y)$ of class $C^{(2N)}$ satisfy $L^Nu = 0$ in D . Writing $a_0 = u$, we seek to construct $2N - 1$ functions $a_0, \dots, a_{2N-1}$ such that $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic in D . In order to do this, we shall introduce three auxiliary functions $\omega_1(x, y)$, $\omega_2(x, y)$, and $\phi(x, y)$; define $a_1(x, y)$ and $a_{2N-1}(x, y)$; and from the three functions a_0, a_1 , and a_{2N-1} deduce the rest. The restriction that D be a rectangle appears in the definition of a_1 and a_{2N-1} .

Recalling that $\int \dots dX^* = \int \dots \sigma_2(x) dx$ and $\int \dots dY = \int \dots (1/\tau_1(y)) dy$, we set

$$(9.2) \quad \omega_1(x, y) \equiv \int_0^y \frac{\partial a_0}{\partial X} dY,$$

$$\omega_2(x, y) \equiv \int_0^x \frac{\partial a_0}{\partial Y^*} dX^*,$$

and

$$(9.3) \quad \phi(x, y) \equiv M^{N-1}(\omega_1 + \omega_2).$$

It will be shown immediately that

$$(9.4) \quad \phi(x, y) = \lambda(x) + \psi(y).$$

Since (9.2) yields

$$\frac{\partial \omega_1}{\partial Y} = \frac{\partial a_0}{\partial X}, \quad \frac{\partial \omega_2}{\partial X^*} = \frac{\partial a_0}{\partial Y^*},$$

we may use the identities

$$\frac{\partial^{2p}}{(\partial Y^* \partial Y)^p} = \frac{\partial}{\partial Y^*} \frac{\partial^{2p-2}}{(\partial Y \partial Y^*)^{p-1}} \frac{\partial}{\partial Y},$$

$$\frac{\partial^{2p}}{(\partial X \partial X^*)^p} \frac{\partial}{\partial X} = \frac{\partial}{\partial X} \frac{\partial^{2p}}{(\partial X^* \partial X)^p},$$

to obtain from (9.2)

$$\begin{aligned} M^{N-1}\omega_1 &= \sum_{k=0}^{N-1} \binom{N-1}{k} \frac{\partial^{2N-2}\omega_1}{(\partial X \partial X^*)^k (\partial Y^* \partial Y)^{N-1-k}} \\ &= \frac{\partial^2}{\partial X \partial Y^*} \sum_{k=0}^{N-2} \binom{N-1}{k} \frac{\partial^{2N-4}a_0}{(\partial X^* \partial X)^k (\partial Y \partial Y^*)^{N-2-k}} \\ &\quad + \int_0^y \frac{\partial^{2N-1}a_0}{\partial X (\partial X^* \partial X)^{N-1}} dY. \end{aligned}$$

Performing a similar simplification on $M^{N-1}\omega_2$, we obtain from (9.3)

$$\begin{aligned} \phi(x, y) &= \frac{\partial^2}{\partial X \partial Y^*} \sum_{k=0}^{N-2} \binom{N-1}{k} \frac{\partial^{2N-4}a_0}{(\partial X^* \partial X)^k (\partial Y \partial Y^*)^{N-2-k}} \\ &\quad + \int_0^y \frac{\partial^{2N-1}a_0}{\partial X (\partial X^* \partial X)^{N-1}} dY \\ &\quad + \frac{\partial^2}{\partial X \partial Y^*} \sum_{k=1}^{N-1} \binom{N-1}{k} \frac{\partial^{2N-4}a_0}{(\partial X^* \partial X)^{k-1} (\partial Y \partial Y^*)^{N-1-k}} \\ &\quad + \int_0^x \frac{\partial^{2N-1}a_0}{\partial Y^* (\partial Y \partial Y^*)^{N-1}} dX^*. \end{aligned}$$

Hence

$$\begin{aligned} \phi_{X^*Y} &= \sum_{k=0}^{N-2} \binom{N-1}{k} \frac{\partial^{2N}a_0}{(\partial X^* \partial X)^{k+1} (\partial Y \partial Y^*)^{N-1-k}} \\ &\quad + \frac{\partial^{2N}a_0}{(\partial X^* \partial X)^N} \\ &\quad + \sum_{k=1}^{N-1} \binom{N-1}{k} \frac{\partial^{2N}a_0}{(\partial X^* \partial X)^k (\partial Y \partial Y^*)^{N-k}} \\ &\quad + \frac{\partial^{2N}a_0}{(\partial Y \partial Y^*)^N} \\ &= \sum_{k=1}^{N-1} \binom{N-1}{k-1} \frac{\partial^{2N}a_0}{(\partial X^* \partial X)^k (\partial Y \partial Y^*)^{N-k}} \\ &\quad + \frac{\partial^{2N}a_0}{(\partial X^* \partial X)^N} \\ &\quad + \sum_{k=1}^{N-1} \binom{N-1}{k} \frac{\partial^{2N}a_0}{(\partial X^* \partial X)^k (\partial Y \partial Y^*)^{N-k}} \end{aligned}$$

$$\begin{aligned}
& + \frac{\partial^{2N} a_0}{(\partial Y \partial Y^*)^N} \\
& = \sum_{k=0}^N \binom{N}{k} \frac{\partial^{2N} a_0}{(\partial X^* \partial X)^k (\partial Y \partial Y^*)^{N-k}} \\
& = L^N a_0 = 0.
\end{aligned}$$

by (9.2) and the hypothesis concerning a_0 . Since $\phi_{X^*Y} = (\tau_1/\sigma_2)\phi_{xy}$ and $\phi_{X^*Y} = 0$, we have $\phi_{xy} = 0$ and (9.3) follows.

Now, let the functions $A(x)$ and $B(y)$ be defined by the equations

$$(9.5) \quad \frac{d^{2N-2}A(x)}{(dXdX^*)^{N-1}} = -\lambda(x),$$

and

$$(9.6) \quad \frac{d^{2N-2}B(y)}{(dY^*dY)^{N-1}} = -\psi(y).$$

It is easily seen that such real functions exist, although they are not uniquely determined.

It is now possible to define $a_1(x, y)$ and $a_{2N-1}(x, y)$. The system $E(\Sigma, N)$

$$(9.7) \quad \begin{aligned} a_{2k,X} &= a_{2k+1,Y}, \\ a_{2k,Y^*} &= a_{2k-1,X^*} - a_{2N-1,X^*} \binom{N}{k}, \end{aligned}$$

($k=0, \dots, N-1$) gives for $k=0$

$$(9.8) \quad \begin{aligned} a_{0,X} &= a_{1,Y}, \\ a_{0,Y^*} &= -a_{2N-1,X^*}, \end{aligned}$$

In order to satisfy (9.8), we define [see (9.2)]

$$(9.9) \quad \begin{aligned} a_1(x, y) &\equiv \omega_1(x, y) + A(x), \\ a_{2N-1}(x, y) &\equiv -\omega_2(x, y) - B(y). \end{aligned}$$

We have that

$$(9.10) \quad M^{N-1}(a_1 - a_{2N-1}) = 0,$$

for, using (9.9)

$$\begin{aligned}
M^{N-1}(a_1 - a_{2N-1}) &= M^{N-1}\omega_1 + \frac{d^{2N-2}A(x)}{(dXdX^*)^{N-1}} \\
&\quad + M^{N-1}\omega_2 + \frac{d^{2N-2}B(y)}{(dY^*dY)^{N-1}} \\
&= \phi - \lambda(x) - \psi(y) \\
&= 0,
\end{aligned}$$

by (9.3) and (9.4).

Since, in view of (9.10), we have one function satisfying $M^{N-1}v = 0$, we may use equation (4.1) "backwards" to produce the missing functions $a_2, \dots, a_{2N-2}$. By the remark made at the beginning of the proof, there exists a $(\Sigma, N-1)$ -monogenic function $g(z) = \sum_{i=0}^{2N-3} \alpha_i j_{N-1}^i$ such that $\alpha_1 = a_1 - a_{2N-1}$. Define $a_{2N-2}(x, y)$ by the relation

$$a_{2N-2} = a_0 - \alpha_0.$$

Since a_{2N-2} and a_{2N-1} are known, we may define a_{2k} and a_{2k+1} for $k=1, \dots, N-2$ by the relations [see equation (4.1)]

$$(9.11) \quad \begin{aligned} a_{2k} &= \alpha_{2k} + \binom{N-1}{k} a_{2N-2}, \\ a_{2k+1} &= \alpha_{2k+1} + \binom{N-1}{k} a_{2N-1}. \end{aligned}$$

Equation (9.11) also holds for $k=0$, and for $k=N-1$ provided α_{2N-2} and α_{2N-1} are taken as zero. We must now show that the a_i thus defined satisfy (9.7). In order to do this, we remark that the α_i satisfy $E(\Sigma, N-1)$

$$(9.12) \quad \begin{aligned} \alpha_{2k,X} &= \alpha_{2k+1,Y}, \\ \alpha_{2k,Y^*} &= \alpha_{2k-1,X^*} - \alpha_{2N-3,X^*} \binom{N-1}{k}, \\ (k=0, \dots, N-2). \end{aligned}$$

First, as has already been pointed out, in view of (9.9), the pair of equations obtained by setting $k=0$ in (9.7) are satisfied. That is

$$(9.13) \quad \begin{aligned} a_{0,X} &= a_{1,Y}, \\ a_{0,Y^*} &= -a_{2N-1,X^*}. \end{aligned}$$

On the other hand, (9.12) gives for $k=0$

$$(9.14) \quad \begin{aligned} \alpha_{0,X} &= \alpha_{1,Y}, \\ \alpha_{0,Y^*} &= -\alpha_{2N-3,X^*}, \end{aligned}$$

and using (9.11) for $k=0$ and $k=N-2$, (9.14) becomes

$$(9.15) \quad \begin{aligned} a_{0,X} - a_{2N-2,X} &= a_{1,Y} - a_{2N-1,Y}, \\ a_{0,Y^*} - a_{2N-2,Y^*} &= -[a_{2N-3,X^*} - a_{2N-1,X^*} \binom{N-1}{N-2}], \end{aligned}$$

which, in view of (9.13), simplifies to

$$(9.16) \quad \begin{aligned} a_{2N-2,X} &= a_{2N-1,Y}, \\ a_{2N-2,Y^*} &= a_{2N-3,X^*} - a_{2N-1,X^*} \binom{N}{N-1}, \end{aligned}$$

the pair of equations obtained from (9.7) by setting $k = N - 1$.

To prove that (9.9) holds for $k = 1, \dots, N - 2$, we first substitute into (9.12) the α_i 's as given by (9.11). The result is

$$(9.17) \quad \begin{aligned} a_{2k,X} - \binom{N-1}{k} a_{2N-2,X} &= a_{2k+1,Y} - \binom{N-1}{k} a_{2N-1,Y}, \\ a_{2k,Y^*} - \binom{N-1}{k} a_{2N-2,Y^*} &= a_{2k-1,X^*} - \binom{N-1}{k-1} a_{2N-1,X^*} \\ &\quad - \binom{N-1}{k} [a_{2N-3,X^*} - \binom{N-1}{N-2} a_{2N-1,X^*}], \end{aligned}$$

($k = 1, \dots, N - 2$). In view of (9.16), (9.17) becomes (9.7). Thus

$f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic and $a_0 = u$ inside the rectangle D .

For convenience, if a domain D is such that Lemma 9.1 holds for D , then we shall call D an adequate domain. To obtain the lemma for a general domain D , we verify the following statements:

(1) If D_1 and D_2 are adequate domains and their intersection $D_1 \cdot D_2$ is a non-vacuous connected set, then their sum $D_1 + D_2$ is also an adequate domain.

(2) If D_1 , D_2 , and D_3 are adequate domains, $D_1 \cdot D_2$ and $D_2 \cdot D_3$ are non-vacuous connected sets, and D_1 and D_3 are mutually exclusive, then $D_1 + D_2 + D_3$ is an adequate domain.

(3) Let a domain be said to be of class n , ($n = 1, 2, \dots$), if its closure is the sum of n mutually exclusive rectangular interiors, plus their boundaries, the rectangular boundaries having their sides parallel to the x and y axes and such that if two of the boundaries have a common part, then the common part contains more than one point. For each n , every domain of class n is adequate.

(4) If a domain D is the sum of a monotone ascending sequence $D_1 \subset D_2 \subset D_3 \subset \dots$ of adequate domains, then D is an adequate domain.

To prove (1), let $f_i(z)$, for $i = 1, 2$, be (Σ, N) -monogenic in D_i and such that $\operatorname{Re} f_i(z) = u$ in D_i . $f_1(z)$ differs from $f_2(z)$ in $D_1 \cdot D_2$ by at most a null function (see Section 8), since $\operatorname{Re}[f_1(z) - f_2(z)] = 0$ in $D_1 \cdot D_2$.

Hence, the function $g(z) \equiv f_1(z) - f_2(z)$ is defined over the whole plane (see Section 8), and the function

$$F(z) = \begin{cases} f_1(z) & \text{in } D_1, \\ f_2(z) + g(z) & \text{in } D_2, \end{cases}$$

is (Σ, N) -monogenic in $D_1 + D_2$ and $Re[F(z)] = u$ there. (2) follows by applying (1) first to D_1 and D_2 and then to $D_1 + D_2$ and D_3 . (3) is proved by induction. A domain of class 1 is the interior of a rectangle; hence, it is adequate. It is easily verified that the closure of a domain of class n is the sum of the closure of a domain of class $n - 1$ and the closure of a domain of class 1. The induction is completed by introducing a suitable rectangle [to play the role of D_2 in (2)] and reasoning as in the proof of (2). (4) follows by applying repeatedly the same argument used in the proof of (1), in order to extend the definition of a (Σ, N) -monogenic function $f_1(z)$ such that $Re f_1(z) = u$ in D_1 to all domains of the ascending sequence.

That Lemma 9.1 holds for any domain D follows from (4), since every domain is known to be the sum of a monotone ascending sequence of domains each of which belongs to class n for some n .

We remark that if D is simply connected, then the (Σ, N) -monogenic function obtained above is single-valued, by the analogue of the monodromy theorem for (Σ, N) -monogenic functions. If D is not simply connected, then the (Σ, N) -monogenic function obtained above may be multiple valued.

From Lemma 9.1, in the same way as was pointed out for $N - 1$ in the course of the proof of that lemma, we have

LEMMA 9.2. *If $v = v(x, y)$ of class $C^{(2N)}$ satisfies the equation $M^N v = 0$ in a domain D , then there exists a (Σ, N) -monogenic function $g(z) = \sum_{i=0}^{2N-1} b_i j_N^i$ such that $b_1 = v$.*

The preceding considerations enable us to give a complete solution to the "inverse problem" for (Σ, N) -monogenic functions by means of the

THEOREM 9.1. *If $u = u(x, y)$ of class $C^{(2N)}$ is such that either (1) $L^N u = 0$ or (2) $M^N u = 0$ in a domain D , then for each integer $k = 0, \dots, N - 1$ there exists (1) a function $f_k(z) = \sum_{i=0}^{2N-1} a_{ki} j_N^i$ which is (Σ, N) -monogenic in D and such that $a_{k,2k} = u$ or (2) a function $g_k(z) = \sum_{i=0}^{2N-1} b_{ki} j_N^i$ which is (Σ, N) -monogenic in D and such that $b_{k,2k+1} = u$.*

We prove only (1), the proof for (2) being analogous. Let u satisfying $L^N u = 0$ be given and let $f_0(z) = \sum_{i=0}^{2N-1} a_{0i} j_N^i$ be the (Σ, N) -monogenic function given by Lemma 9.1, with $a_{00} = u$. Then $f_k(z) = \sum_{i=0}^{2N-1} a_{ki} j_N^i = (1/K_{N,k}) f_0(z)$, $k = 0, \dots, N-1$ are the desired functions, where the $K_{N,k}$ are given in Theorem 2.7. By Theorem 2.7, $a_{k,2k} = a_{00} = u$. By Theorem 2.8 and Corollary 6.1, it follows that the $f_k(z)$ are (Σ, N) -monogenic.

Finally, we remark that to obtain the most general (Σ, N) -monogenic function $F_k(z) = \sum_{i=0}^{2N-1} A_{ki} j_N^i$ such that $A_{k,2k} = u$, we merely add to the $f_k(z)$ the appropriate "null functions" from 8.

10. Formal power series and the expansion theorem. An expression of the form

$$(10.1) \quad \sum_{n=0}^{\infty} \alpha_n \cdot Z^{(n)}(z_0; z),$$

where $\alpha_n = \sum_{i=0}^{2N-1} \alpha_{ni} j_N^i$ are hypercomplex constants, will be called a formal power series. Our primary objective in this section is to show that if $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic at a point z_0 , then it can be represented uniquely by a formal power series (10.1) in a sufficiently small neighborhood of z_0 . Preliminary to this, we shall establish the existence, under certain conditions on the coefficients α_n , of a domain of convergence of (10.1) and also show that a (Σ, N) -monogenic function in a domain D is uniquely determined by the value of the function and of all its (Σ, N) - and (Σ', N) -derivatives at a point of D .

In 3 the higher (Σ, N) -derivatives $f^{[n]}(z) = \sum_{i=0}^{2N-1} a_i^{[n]} j_N^i$ of a (Σ, N) -monogenic function $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ were defined, and it was shown that $f^{[2n]}(z)$ is (Σ, N) -monogenic and $f^{[2n+1]}(z)$ is (Σ', N) -monogenic.

Equation (3.5) gives that $f^{[1]}(z) = \frac{d_{\Sigma} f}{d_{\Sigma} z} = \sum_{i=0}^{2N-1} a_i^{[1]} j_N^i$, where

$$(10.2) \quad \begin{aligned} a_{2k}^{[1]} &= a_{2k, X} = \sigma_1 a_{2k, x}, \\ a_{2k+1}^{[1]} &= a_{2k+1, X^*} = (1/\sigma_2) a_{2k+1, x}, \end{aligned}$$

($k = 0, \dots, N-1$). Repeated application of (10.2) gives immediately

$$(10.3) \quad \begin{aligned} a_{2k}^{[2n]} &= \frac{\partial^{2n} a_{2k}}{(\partial X^* \partial X)^n}, & a_{2k+1}^{[2n]} &= \frac{\partial^{2n} a_{2k+1}}{(\partial X \partial X^*)^n}, \\ a_{2k}^{[2n+1]} &= \frac{\partial^{2n+1} a_{2k}}{\partial X (\partial X^* \partial X)^n}, & a_{2k+1}^{[2n+1]} &= \frac{\partial^{2n+1} a_{2k+1}}{\partial X^* (\partial X \partial X^*)^n}, \end{aligned}$$

($n = 0, 1, \dots$), ($k = 0, \dots, N-1$).

We shall employ a lemma given by Bers and Gelbart (1)

LEMMA 10.1. *Let $F(x)$, $G(x)$, $H(x)$ be analytic functions of the real variable x in the neighborhood of $x = 0$. Define*

$$(10.4) \quad F_0(x) = F(x),$$

$$F_n(x) = \begin{cases} G(x) \frac{dF_{n-1}}{dx} & \text{if } n \text{ is odd,} \\ H(x) \frac{dF_{n-1}}{dx} & \text{if } n \text{ is even.} \end{cases}$$

Then there exists a constant $C > 0$ such that

$$(10.5) \quad |F_n(0)| < n! C^n,$$

for all n .

The following lemma will be used in the proof of the expansion theorem.

LEMMA 10.2. *If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic at $z = z_0$, then there exists a constant $C > 0$ such that*

$$(10.6) \quad \|f^{[n]}(z_0)\| < n! C^n,$$

for all n .

Proof. Without loss in generality z_0 may be taken to be zero. Applying Lemma 10.1 for each $k = 0, \dots, N-1$, with

$$F(x) = a_{2k}(x, 0), \quad G(x) = \sigma_1(x), \quad H(x) = 1/\sigma_2(x),$$

and using (10.3), we see that there exist N constants C_{2k} such that

$$(10.7) \quad |a_{2k}^{[n]}(0, 0)| < n! C_{2k}^n,$$

($k = 0, \dots, N-1$).

Similarly, there exist N constants C_{2k+1} such that

$$(10.8) \quad |a_{2k+1}^{[n]}(0, 0)| < n! C_{2k+1}^n,$$

But, by (2.24) and (2.25)

$$(10.9) \quad \|f^{[n]}(0)\| \leq \sum_{i=0}^{2N-1} |a_i^{[n]}(0,0)| \|j_N^i\|,$$

and $\|j_N^i\| = 1$ for $0 \leq i \leq 2N-1$, so that from (10.7) and (10.8)

$$(10.10) \quad \|f^{[n]}(0)\| < n! \left(\sum_{i=0}^{2N-1} C_i \right)^n.$$

THEOREM 10.1. If $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic at $z = z_0$ and

$$(10.11) \quad f^{[n]}(z_0) = 0, \quad n = 0, 1, \dots,$$

then

$$f(z) \equiv 0.$$

The proof is by induction on N . For $N = 1$ the theorem has been proved by Bers and Gelbart (1). Suppose that the conclusion is true for $N-1$, and let $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ be a (Σ, N) -monogenic function satisfying (10.11) at $z = z_0$. Equation (10.11) implies that

$$(10.12) \quad a_i^{[n]}(z_0) = 0, \quad n = 0, 1, \dots, \\ (i = 0, \dots, 2N-1).$$

By Lemma 4.1, the function $\phi(z) = \sum_{i=0}^{2N-3} \alpha_i j_{N-1}^i$, where

$$(10.13) \quad \alpha_{2k} = a_{2k} - \binom{N-1}{k} a_{2N-2}, \\ \alpha_{2k+1} = a_{2k+1} - \binom{N-1}{k} a_{2N-1},$$

($k = 0, \dots, N-2$), is $(\Sigma, N-1)$ -monogenic in a neighborhood of z_0 . From (10.11), we have that

$$(10.14) \quad \phi^{[n]}(z_0) = 0, \quad n = 0, 1, \dots,$$

since (10.13) and (10.3) together give

$$(10.15) \quad \alpha_{2k}^{[n]}(z_0) = a_{2k}^{[n]}(z_0) - \binom{N-1}{k} a_{2N-2}^{[n]}(z_0) = 0, \\ \alpha_{2k+1}^{[n]}(z_0) = a_{2k+1}^{[n]}(z_0) - \binom{N-1}{k} a_{2N-1}^{[n]}(z_0) = 0,$$

($k = 0, \dots, N-2$), using (10.12).

By (10.14) and the induction hypothesis, we have that $\phi(z) = \sum_{i=0}^{2N-3} \alpha_i j_{N-1}^i \equiv 0$, and from (10.13)

$$(10.16) \quad \begin{aligned} a_{2k} &= \binom{N-1}{k} a_{2N-2}, \\ a_{2k+1} &= \binom{N-1}{k} a_{2N-1}, \end{aligned}$$

($k = 0, \dots, N-2$). Since $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N) -monogenic, then

$$(10.17) \quad \begin{aligned} a_{2k, X} &= a_{2k+1, Y}, \\ a_{2k, Y^*} &= a_{2k-1, X^*} - a_{2N-1, X^*} \binom{N}{k}, \end{aligned}$$

($k = 0, \dots, N-1$). Replacing a_{2k} and a_{2k+1} in (10.17) by their values from (10.16) yields

$$(10.18) \quad \begin{aligned} a_{2N-2, X} &= a_{2N-1, Y}, \\ a_{2N-2, Y^*} &= -a_{2N-1, X^*}. \end{aligned}$$

Replacing a_{2N-2} and a_{2N-1} in (10.18) by their values from (10.16) gives

$$(10.19) \quad \begin{aligned} a_{2k, X} &= a_{2k+1, Y}, \\ a_{2k, Y^*} &= -a_{2k+1, X^*}, \end{aligned}$$

($k = 0, \dots, N-1$). This last equation shows that the N functions

$$(10.20) \quad f_k(z) = a_{2k} + j_1 a_{2k+1},$$

($k = 0, \dots, N-1$), are $(\Sigma, 1)$ -monogenic. But from (10.12) and (10.3), we have that

$$(10.21) \quad f_k^{[n]}(z_0) = 0, \quad n = 0, 1, \dots$$

($k = 0, \dots, N-1$). Hence, $a_i = 0$ for $i = 0, \dots, 2N-1$ and $f(z) \equiv 0$.

Returning to the consideration of the formal power series (10.1), we remark that such a series represents a (Σ, N) -monogenic function in any domain in which it converges uniformly, and may be (Σ, N) -integrated term by term in any such domain. We shall show that a domain of uniform convergence of (10.1) is determined by the sequence of coefficients α_n .

THEOREM 10.2. Let $\alpha_n = \sum_{i=0}^{2N-1} \alpha_n i j_N^i$ be a sequence of numbers of A_{2N} and $z = z_0$ be fixed. If

$$(10.22) \quad \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{\|\alpha_n\|}} = R,$$

does not vanish, then there exists a neighborhood D of z_0 such that the formal power series

$$(10.23) \quad f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot Z^{(n)}(z_0; z),$$

and all the formal power series obtained by successively (Σ, N) - and (Σ', N) -differentiating (10.23) term by term converge uniformly and absolutely in D .

Proof. Let D be the domain

$$(10.24) \quad |z - z_0| < \frac{\rho}{2^{\frac{1}{2}} \Gamma_N m_{\Sigma} (|z_0| + R)}, \quad 0 < \rho < R.$$

From (10.22), it follows that D has the required property, for the inequality (7.14) implies

$$\sum_{n=0}^M \|\alpha_n \cdot Z^{(n)}(z_0; z)\| < 2 \Gamma_N \sum_{n=0}^M \|\alpha_n\| \rho^n,$$

and similar inequalities hold for the derived series.

Successive termwise (Σ, N) - and (Σ', N) -differentiation of (10.23) and evaluation at z_0 of the resulting formal power series gives the "Taylor formulas"

$$(10.25) \quad f^{[n]}(z_0) = n! \alpha_n,$$

$n = 0, 1, \dots$

THEOREM 10.3. Let $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ be (Σ, N) -monogenic at $z = z_0$.

Then $f(z)$ can be (uniquely) expanded at $z = z_0$ in a formal power series of the form (10.23).

Proof. Let the sequence of hypercomplex numbers α_n be defined by (10.25). By virtue of Lemma 10.2

$$\lim \sqrt[n]{\|\alpha_n\|} \leq C.$$

Hence, by Theorem 10.2, the formal power series

$$g(z) = \sum_{n=0}^{\infty} \alpha_n \cdot Z^{(n)}(z_0; z),$$

converges uniformly and absolutely in a neighborhood of z_0 and

$$g^{[n]}(z_0) = n! \alpha_n,$$

$n = 0, 1, \dots$. Then $f(z) - g(z) \equiv 0$ by Theorem 10.1.

Theorems 10.1 and 10.3 serve to show that the class of functions $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ which can be expanded in a formal power series of the form (10.1) in a neighborhood of z_0 coincides with the class of functions which are (Σ, N) -monogenic at z_0 according to Definition 3.1. This corresponds to the equivalence of the Cauchy and Weierstrass definitions of an analytic function.

11. Complete systems of particular solutions of $L^N u = 0$. We define

$$(11.1) \quad U_{Nnv}(x_0, y_0; x, y) = \operatorname{Re}[j_N^v \cdot Z^{(n)}(z_0; z)],$$

where $z = x + iy$, $z_0 = x_0 + iy_0$, $n = 0, 1, \dots, v = 0, 1, \dots, 2N-1$. Obviously, each U is a (real-valued) particular solution of the equation $L^N u = 0$, analytic in the whole plane. Notice that for $n < N$ some of the functions (11.1) vanish. The sequence $\{U_{Nnv}\}$ has the completeness properties mentioned in Section 1. We have

THEOREM 11.1. *Each solution $u = u(x, y)$ of the equation $L^N u = 0$, which is of class $C^{(2N)}$ in a domain containing the point z_0 , admits the expansion*

$$(11.2) \quad u(x, y) = \sum_{n=0}^{\infty} \sum_{v=0}^{2N-1} a_{nv} U_{Nnv}(x_0, y_0; x, y),$$

where the a 's are real constants and the series converges uniformly and absolutely in some neighborhood of z_0 .

Proof. By Lemma 9.1, there exists, in a suitable neighborhood of z_0 , a (Σ, N) -monogenic function $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ such that $a_0 = u$. By Theorem 10.3, we have

$$(11.3) \quad f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot Z^{(n)}(z_0; z),$$

where

$$\alpha_n = \sum_{v=0}^{2N-1} a_{nv} j_N^v,$$

$n = 0, 1, \dots$, in a sufficiently small neighborhood of z_0 . (11.2) follows upon equating the real parts of both sides of (11.3), using (7.8) and (11.1).

The sequence of real functions (11.1) depends upon the point z_0 . In order to prove a more general expansion theorem, we establish

THEOREM 11.2. *Let z_0 and b be points of the plane and let α be a number of A_{2N} , then*

$$(11.4) \quad \alpha \cdot Z^{(n)}(z_0; z) = \sum_{k=0}^n \binom{n}{k} A_k \cdot Z^{(n-k)}(b; z),$$

where

$$(11.5) \quad A_k = \begin{cases} \alpha \cdot Z^{(k)}(z_0; b) & \text{if } k \text{ is even,} \\ \alpha \cdot \tilde{Z}^{(k)}(z_0; b) & \text{if } k \text{ is odd.} \end{cases}$$

Equation 11.4 is another analogue of the binomial formula. This theorem follows immediately from the expansion theorem 10.3, with the aid of (7.3) and (10.25). We note that since (11.4) is a relation between iterated integrals of the functions σ_i and τ_i , the equation holds independently of the expansion theorem and of the analyticity of the σ_i and τ_i .

We now define

$$(11.6) \quad U_{Nnv}(x, y) = U_{Nnv}(0, 0; x, y).$$

From Theorem 11.2, taking the origin as b , we obtain from (11.1) and (11.6)

$$(11.7) \quad U_{Nnv}(x_0, y_0; x, y) = \sum_{m=0}^n \sum_{\mu=0}^{2N-1} b_{Nnv m \mu} U_{Nm \mu}(x, y),$$

where $n = 0, 1, \dots; v = 0, 1, \dots, 2N-1$, and the b 's are real constants. Hence, we have the

THEOREM 11.3. *Each solution $u = u(x, y)$ of the equation $L^N u = 0$, which is of class $C^{(2N)}$ in a domain containing the point z_0 , admits the expansion*

$$(11.8) \quad u(x, y) = \sum_{n=0}^{\infty} \left\{ \sum_{v=0}^{2N-1} \sum_{m=0}^n \sum_{\mu=0}^{2N-1} c_{Nnv m \mu} U_{Nm \mu}(x, y) \right\},$$

where the c 's are real constants and the series converges uniformly and absolutely in some neighborhood of z_0 .

Finally, we observe that by (7.4), (7.5), and (7.10) to (7.13), the U_{Nnv} are linear combinations with real constant coefficients of the functions given in (7.9), which are independent of N .

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